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Instantons.

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Whilst there is no doubt that non-abelian gauge theories form the basis, in one way or another, of our present understanding of gravitation, strong, weak and electromagnetic interactions, not much is known about the mathematical structure of the associated field theories. It is possible that questions relating to quark confinement or the Higgs mechanism would become accessible mathematically if we had a good understanding of gauge theories which went beyond perturbation theory. That seems to us sufficient reason to embark upon the exploration of many different aspects of the theory even though the outcome is not assured; such as instantons, the 1/N expansion or lattice gauge theories.

This abstract is intended to indicate the content of five lectures, one of which was introductory whilst the others described in some detail most of the ingredients that enter into a calculation of the contribution of instantons to the gauge theory functional integral.[(1),(2)]

A typical gauge theory is defined by an action such as,

$$S = \int d^4x \left[-\tfrac{1}{2}\, \mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu}) + i\,\bar{\Psi}\,\gamma\cdot D\,\Psi \right] \tag{1}$$

where ψ is a fermi field transforming according to some representation of the gauge group G, and $F_{\mu\nu}$ is the field strength tensor regarded as an (anti-hermitean) element of the Lie algebra of G in the adjoint representation. Thus, if A_μ is the vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2}$$

while for example, for ψ in the fundamental representation of G, the covariant derivative is

$$\gamma . D\psi = \gamma^\mu(\partial_\mu + A_\mu)\psi \tag{3}$$

Under gauge transformations

$$\begin{aligned} \psi &\to g^{-1}\psi \\ A_\mu &\to g^{-1}A_\mu g + g^{-1}\partial_\mu g \qquad\qquad g(x) \in G \\ F_{\mu\nu} &\to g^{-1}F_{\mu\nu}g \end{aligned} \tag{4}$$

and the action is, of course, invariant.

The Green functions of the quantum field theory corresponding to the action (1) are given by a set of functional integrals of the form,

$$Z_\Phi = Z_1^{-1}\int d[A_\mu]\, d[\bar\psi]\, d[\psi]\, e^{iS/g^2}\, \Phi(A, \bar\psi, \psi) \tag{5}$$

where Φ represents products of the fields at different space-time points. Belavin, Polyakov, Schwarz and Tyupkin[2] initiated a program of exploration for the functional integral (5) (ignoring the fermi field ψ) by making the following observations:

(i) to define the functional integral properly it should be defined instead for a euclidean space-time, in which case the action S is positive (or zero) and the i in the exponent is replaced by -1,

(ii) then, whenever S is finite the potential A_μ must approach at large distances a pure gauge $g^{-1}\partial_\mu g$. The gauge function g (which is to be regarded as a function of the angles on the 'sphere at ∞' in R^4) ~~can be thought of as a map from the sphere S^3~~ into the gauge group G. Such maps fall naturally into equivalence classes under homotopy and, if the group G is simple and compact the equivalence classes are labelled y the integers. Moreover, ignoring ψ , the action obeys the following inequality (provided we choose A_μ to belong to a specific homotopy class in the above sense)

$$S = -\tfrac{1}{4}\int d^4x \ \mathrm{tr}\left(F_{\mu\nu} \mp {}^*F_{\mu\nu}\right)\left(F_{\mu\nu} \mp {}^*F_{\mu\nu}\right) \mp \tfrac{1}{2}\int d^4x \ \mathrm{tr}\left(F_{\mu\nu}\,{}^*F_{\mu\nu}\right)$$
$$\geq -\tfrac{1}{2}\int d^4x \ \mathrm{tr}\left(F_{\mu\nu}\,{}^*F_{\mu\nu}\right) \equiv 8\pi^2|k| \tag{6}$$

where the integer k is just the label for the class to which the potential belongs.

(iii) The equality in eq.(6) is attained if the field strength $F_{\mu\nu}$ is self-dual (or anti-self-dual) i.e.,

$${}^*F_{\mu\nu} \equiv \tfrac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = \pm F_{\mu\nu} \tag{7}$$

The problem of finding all the solutions to eq.(7) for a given integer k is of interest in its own right and has exercised the ingenuity of a number of physicists and attracted the attention of mathematicians. The latter were able to cast the problem in terms of algebraic geom tr and hence find a remarkable way of solving it. The solution to the pr blem provided by Atiyah, Drinfeld, Hitchin and Manin (ADHM)[4], whilst r l ing on deep mathematics for its derivation, can nevertheless b pre t ed in a simple and appealing form and used to tackle the problem of how to estimate the functional integral (5).

The ADHM construction works for any gauge group G and is described in more detail elsewhere[4,5,6]. However, for SU(2) we can quickly summarise the results. Writing

$$A_\mu = v^+\partial_\mu v \ , \quad v^+v = 1 \tag{8}$$

where v is a matrix with k+1 rows and 1 column (whose entries are 2x2 matrices of the form $\alpha = \alpha_0 - i\underline{\alpha}.\underline{\sigma}$ where $\underline{\sigma}$ are the Pauli σ-matrices (i.e. quaternions)), the vector potential solves equation(7) and belongs to the class labelled by k provided the components of v are chosen in a clever way. Moreover, all solutions to eq(7) may be found in this manner.

The matrix v is to be chosen as follows. v is constrained to be orthogonal to a set of k other similar matrices which are themselves linearly independent and linear functions of the euclidean coordinates x_μ, (also best thought of for this purpose as the 2x2 matrix $X_0 - i\underline{X}.\underline{\sigma}$). Thus

$$V^+(a+bx) = 0 \tag{9}$$

or, making explicit all the indices and summations,

$$\sum_{i,\beta} (V^+)_{i\alpha\beta} \left(a_{ij\beta\gamma} + \sum_\delta b_{ij\beta\delta} X_{\delta\gamma} \right) = 0 \qquad \begin{array}{l} \alpha,\beta,\gamma,\delta = 1,2 \\ i = 1\dots k+1 \\ j = 1\dots k \end{array}$$

The parameters describing the solutions to eq(7) (instantons) reside in the constant matrices a and b which have to be such that,

$$\left[(a+bx)^+ (a+bx) \right]_{ij\alpha\beta} = \delta_{\alpha\beta}\, v_{ij} \;, \; \forall x \text{ and } \begin{array}{l} \alpha,\beta = 1,2 \\ i,j = 1\dots k \end{array} \tag{10}$$

Unfortunately, the constraints (10) have not so far proved to be soluble and the degrees of freedom of the instantons ($8|k|-3$ of them) cannot be made completely explicit. However, even without the explicit representation for the solutions it is possible to go some way towards evaluating the functional integral (5) as $g \to 0$. The reason for this is that certain useful quantities, such as Green functions and functional determinants (for differential operators like the gauge covariant Laplacian D^2), are calulable albeit as implicit functions of the instanton parameters.

Writing the euclidean version of (5) and omitting the fermi field we may write an asymptotic expansion for the functional integral valid for small g,

$$Z_\Phi \propto \sum_k \left(\frac{\mu}{\sqrt{2\pi}\, g}\right)^{N(k)} e^{-\frac{8\pi^2|k|}{g^2}} \int \prod_1^{N(k)} dt_i \, \frac{\det'(-D^2/\mu^2)}{[\det'(-\Delta_1/\mu^2)]^{1/2}} \sqrt{N}\; \Phi \qquad (11)$$

In eq(11) N(k) is the number of degrees of freedom of a k-instanton, μ is a parameter with the dimensions of an inverse length and, t_i, i=1..N(k) are the instanton parameters. The instanton parameters occur in the integrands implicitly, since D^2 is the gauge covariant Laplacian (evaluated in the adjoint representation of the gauge group) and

$$(\Delta_1)_{\mu\nu} = D^2\delta_{\mu\nu} + 2[F_{\mu\nu},\ ], \qquad (12)$$

both evaluated for the instanton vector potential. The primes on the determinants indicate the omission of zero modes and the factor $\sqrt{N}$ is given by

$$N = \det\left[\int d^4x \, \frac{\partial A_\mu}{\partial t_i}\frac{\partial A_\mu}{\partial t_j}\right] \qquad (13)$$

A nice derivation of equation (11) has been given by Schwarz (7) but it can also be thought of as the result of the familiar Fadeev-Popov manipulation in the 'background' gauge.

Several problems with the quantities appearing in eq(11) are immediately apparent. The determinants have to be defined in a sensible way; there is no point merely taking the product of eigenvalues, as one would for a finite dimensional matrix. There are two reasons for this. Firstly, the operators are not defined over a compact manifold and hence there are 'infra-red' divergences--simply because the differential operators have eigenvalues arbitrarily close to zero. On the other hand there are also arbitrarily large eigenvalues leading to 'ultra-violet' divergences. The former are not so serious as the latter since they may be removed by performing all calculations on the compact manifold S^4 (regarded as the surface of a five-dimensional sphere of large radius R). It then transpires that the divergent parts of the determinants as $R \to \infty$ are independent of the instanton parameters and contribute an overall factor to $Z_{\frac{1}{2}}$, eq(11), rather than contributing differently term by term. (8) In particular, the variation of the logarithm of any determinant with respect to any instanton parameter will be finite as $R \to \infty$.

The ultraviolet divergences are more serious and require more skill for their removal. There are many ways of discussing ultraviolet divergences developped by field theorists but a particularly elegant, and for this case very useful one is the so-called zeta function regularisation. (9) It has been studied by many people in various contexts over the years, and recently advocated by Hawking in the context of general relativity. (10) The method also corresponds closely to the way in which mathematicians have decided to define the 'torsion' of an operator such as D^2 (or, such as the Laplacian on a Riemannian manifold studied by Ray and Singer). (11)

Some details of the zeta function method of defining determinants were explained and illustrated by examples from quantum mechanics and field theory in addition to being applied to the gauge theory problem. Briefly, the ideas are as follows.

For a finite dimensional hermitean matrix A whose eigenvalues are all positive (not zero), we may define a 'zeta function'

$$\zeta_A(s) = \sum_{1}^{\dim A} \lambda_n^{-s} \qquad (14)$$

with the following obvious properties

(a) $\dim A = \zeta_A(0)$ (15)

(b) $\det A = \exp\left(-\frac{d\zeta_A}{ds}\Big|_{s=0}\right)$. (16)

For finite dimensional matrices the zeta function (14) is an analytic function of the complex variable s. For operators, such as $\left(-D^2+\frac{2}{R^2}\right) = \mathcal{D}$

defined over S^4, the corresponding zeta function has to be defined by analytic continuation since the definition as an infinite sum, analogous to eq(14), is convergent, typically, only for sufficiently large Res, (in our case Res>2). So defined $\zeta_{\mathcal{D}}(s)$ is an analytic function of s with simple poles for certain real positive values of s (in our case s=1,2)$^{(12)}$ In particular, $\zeta_{\mathcal{D}}(s)$ is regular at s=0.

To perform the analytic continuation it is convenient to use an integral representation for $\zeta_{\mathcal{D}}(s)$:

$$\zeta_{\mathcal{D}}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} \int tr\left(\mathcal{G}(x,x;t) \right) d^4x \tag{17}$$

where $\mathcal{G}(x,y;t)$ is the solution to the equation

$$\mathcal{D}_x\, \mathcal{G}(x,y;t) = \frac{\partial \mathcal{G}(x,y;t)}{\partial t} \quad ; \quad \mathcal{G}(x,y,0) = \delta(x-y). \tag{18}$$

Thus, in our case (and in the limit $R \to \infty$)

$$\begin{aligned} \zeta_{\mathcal{D}}(0) &= \underset{s=0}{\mathrm{Res}} \int_0^{\infty} dt\, t^{s-1} \int d^4x\; tr\, \mathcal{G}(x,x;t) \\ &= \frac{1}{16\pi^2}\, \frac{1}{12} \int d^4x\; tr\left(F_{\mu\nu}F_{\mu\nu}\right) = -\frac{k\,C(A)}{6} \end{aligned} \tag{19}$$

where C(A) is the value of the quadratic casimir operator for the gauge group G in the adjoint representation, (i.e. for SU(N), C(A)=N). On the other hand for the operator Δ_1

$$\zeta_{\Delta_1}(0) = \frac{10}{3}\, k\, C(A) - N(k) \tag{20}$$

just as it should be bearing in mind what we already know about asymptotic freedom, the value of N(k) and the fact that (as $R \to \infty$)

$$\left[\det{}'\left(-\frac{D^2}{\mu^2}\right)\right]^4 = \det{}'\left(-\frac{\Delta_1}{\mu^2}\right)$$

For the details of all the foregoing remarks ref(2) might be found helpful. Certainly, in the sense that the genera lisations of eqs(15) and (16) determine the dimensions of the differential operators in which we are interested, and hence the scaling properties of their determinants, we can verify the relationship between a redefinition of μ and the runnin coupling constant.

To evaluate the determinants is trickier. An examination of the above manipulations leading to $\zeta_{-D^2}(0)$, $\zeta_{\Delta_1}(0)$ reveal that they work because these quantities are the residues of the pole at s=0 in the analytic continuation of the integral in eq(17). For the determinants that is not the case. However, if instead of the determinant we consider its variation with respect to the set of instanton parameters then a similar, but rather more complicated, calculation can be performed. Thus, in more detail:

$$\delta\zeta'_{-D^2}(0) = \operatorname*{Res}_{s=0} \int_0^\infty dt\, t^{s-1} \int d^4x\, d^4y\ \mathrm{tr}\left(\mathcal{G}(x,y;t)\,\delta D^2\, G(y,x)\right) \tag{21}$$

where G(x,y) is the Green function for the covariant Laplacian D^2,

$$D_x^2\ G(x,y) = -\delta(x-y) \tag{22}$$

Brown and Creamer(12) pointed out that whenever the gauge vector potential, used to define the Green function, etc., above, satisfies the sourceless Yang-Mills equations it will be possible to split the Green function into two pieces, only one of which is singular, viz.,

$$G(x,y) = \frac{1}{4\pi^2|x-y|^2}\, P\exp\int_x^y A\cdot dx \ + R(x,y) \tag{23}$$

In which case a careful calculation shows that(13,8)

$$\delta\zeta'_{-D^2}(0) = \int d^4x\ \mathrm{tr}\left[\delta A_\mu\left(\vec{D}_\mu R(x,y) + R(x,y)\overleftarrow{D}_\mu\right)_{x=y}\right] \tag{24}$$

the manifestly singular parts in eq(23) being automatically excluded.

In order to proceed further it is necessary to compute all the constituents of eq(24) in some detail. It has already been pointed out how the vector potential A_μ is constructed from the (oblong) matrix v, via eqs(8),(9) and (10). Varying the instanton parameters amounts to varying the components of the matrices a and b whilst maintaining the constraints expressed by eq(10). Thus D_μ and δA_μ are known. The quantity R(x,y) would be calculable given G(x,y) and a way of calculating the path-ordered exponential from the vector potential. The latter is possible (up to any required order in $(x-y)_\mu$) but somowhat tedious since we are unaware of any simple expression for it. Fortunately, the former is also possible, and perhaps amazingly, there is quite a simple and elegant expression for the Green function.(5,6)

The basic result about instanton Green functions is that if we consider the covariant Laplacian in the fundamental representation of the gauge group then its Green function $G_F(x,y)$ is given by

$$G_F(x,y) = \frac{v^+(x)\,v(y)}{4\pi^2|x-y|^2} \tag{25}$$

In other words, the simplest generalisation away from the Green function for the ordinary Laplacian, ∂^2 , having the correct gauge transformation properties and involving the matrix v. However, to evaluate eq(24) we need the Green function for the Laplacian in the adjoint representation of the gauge group, $G_A(x,y)$. To understand how the Green function we require relates to the one we know, eq(25), it is necessary to understand the formation of tensor products within the context of the ADHM construction, along the lines proposed in refs(14 or 15). Needless to say, whilst we may regard the adjoint representation of any group as (part of) the tensor product representation $F\otimes F^*$, and hence write the vector potential simply in terms of v, it is too much to expect to be able to take the tensor product of quantities like those appearing in eq(25) to produce the adjoint Green function. Rather the correct procedure leads to [14]

$$G_{f\otimes f^*} = \frac{1}{4\pi^2|x-y|^2}\, v^+(x)\otimes v^T(x)\,(1-\mathcal{M})\,v(y)\otimes v^*(y) \qquad (26)$$

where $\mathcal{M}$ is an interesting matrix quantity, depending upon the instanton parameters a,b in a completely conformally invariant way. The Green function for the adjoint representation can be deduced from eq(26) by projection.

Armed with expressions for A_μ , δA_μ , eqs(25) and (26) and information regarding tensor products it is possible to derive expressions for $\delta\Sigma'_{-D^2}(0)$, and hence for the determinant of $-D^2/\mu^2$, in any representation of the gauge group. This has been done (with a varying degree of completeness) by a number of groups [5,16,17] and for the adjoint representation by Jack.[18] The final tasks of undoing the variation with respect to the instanton parameters and writing the result in a useful form has also been attempted, the most complete results so far being those of Jack.[18] However, unlike the situation with the O(3) σ-model in two dimensions,[19] it has not yet proved possible to recognise the terms in the expansion (11) as contributions to a known partition function.[20] In view of that, a great deal of work remains to be done before any useful physical information can be extracted from (11). Besides, it is possible that other natural and interesting structures will eventually be revealed and have to be taken into account in estimating the functional integral.

To conclude we shall summarise some of the most recent results concerning the determinants.[18] Basically, they relate determinants constructed for tensor product instantons to determinants constructed from the factors in the tensor product. For example, for SU(2) we have (for the tensor product of two 2 dimensional representations)

$$\ln \mathrm{Det}(-D^2_{(1)}) = -6\ln \mathrm{Det}(-D^2_{(2)}) - \ln \mathrm{Det}\,(M_S\, v\otimes v)$$

$$+ \frac{1}{32\pi^2}\int d^4x\, \ln \det(f v)\, \partial^2\partial^2 \ln \det(f v) + \text{const.}$$

$$\ln \mathrm{Det}\left(-D^2_{(3)}\right) = 10 \ln \mathrm{Det}\left(-D^2_{(2)}\right) - \ln \mathrm{Det}\left(M_A\ \nu \otimes \nu\right)$$
$$+ \frac{1}{32\pi^2} \int d^4x \ \ln \partial \mathrm{et}(f\nu)\, \partial^2\partial^2 \ln \partial \mathrm{et}(f\nu) + \text{constant}.$$

where the bracketed subscript refers to the dimension of the SU(2) representation, M_A , M_S are the antisymmetric and symmetric parts of a matrix related to $\mathfrak{M}$ (eq(26), $\nu = b^+ b$ (b as in eqs(9)and(10)) and $f = \left[(a+bx)^+ (a+bx)\right]^{-1}$. But, since the one dimensional representation is the trivial one we have:

$$\ln \mathrm{Det}\left(-D^2_{(1)}\right) = -\frac{1}{6} \ln \mathrm{Det}\left(M_S\ \nu \otimes \nu\right)$$
$$+ \frac{1}{192\pi^2} \int d^4x \ \ln \partial \mathrm{et}(f\nu)\, \partial^2\partial^2 \ln \partial \mathrm{et}(f\nu) + \text{const.} \qquad (27)$$

and

$$\ln \mathrm{Det}\left(-D^2_{(3)}\right) = -\frac{5}{3} \ln \mathrm{Det}\left(M_S\ \nu \otimes \nu\right) - \ln \mathrm{Det}\left(M_A\ \nu \otimes \nu\right)$$
$$+ \frac{1}{12\pi^2} \int d^4x \ \ln \partial \mathrm{et}(f\nu)\, \partial^2\partial^2 \ln \partial \mathrm{et}(f\nu) + \text{const.} \qquad (28)$$

the latter being the desired result for the adjoint representation of SU(2). The quantities appearing on the right hand sides of eqs(27) and (28) have not yet been explicitly evaluated. However, since it is possible to form any given representation by tensor products in several different ways (e.g. $(2 \otimes 2)_S = (3)$, $(3 \otimes 3)_A = (3)$) there may be useful relationships between different integrals involving the instanton parameters. Exploiting these relations may lead to more useful formulae than (27) and (28).

References.

For reviews of some of the topics mentioned in these lectures and for a more comprehensive set of references, refs. 1 and 2 below may be useful

1. E.Corrigan , contribution to Phys. Reports 49C (1979) 95.
2. E.Corrigan and P.Goddard, lectures given at the 1979 Canadian Mathematical Society meeting on Geometrical and Topological Methods in Gauge Theories held at McGill University, DAMTP preprint 79/18
   C.Bernard, UCLA preprint 79/TEP/12.
3. A.A.Belavin, A.M.Polyakov, A.S.Schwarz and Yu.S.Tyupkin, Phys. Letts. 59B (1975) 85.
4. M.F.Atiyah, V.G.Drinfeld, N.J.Hitchin and Yu.I.Manin, Phys. Letts. 65A (1978) 185.
   V.G.Drinfeld and Yu.I.Manin, Comm. Math. Phys. 63 (1978) 177.
5. E.Corrigan, D.B.Fairlie, P.Goddard and S.Templeton, Nucl. Phys.B140 (1978) 31.
6. N.H.Christ, N.K.Stanton and E.J.Weinberg, Phys. Rev. D18 (1978) 2013.
7. A.S.Schwarz, Lett. Math. Phys. 2 (1978) 201, 247; Comm. Math. Phys. 64 (1979) 233, 67 (1979) 1.
8. E.Corrigan, P.Goddard, H.Osborn and S.Templeton, Nucl. Phys. B159 (1979)4
   H.Osborn, Nucl. Phys. B159 (1979) 497.
9. See ref 2. for references on this.
10. S.Hawking, Comm. Math. Phys. 55 (1977) 133.
11. D.B.Ray and I.M.Singer, Adv.Math. 7 (1971) 145.
12. See for example P.B.Gilkey, The Index Theorem and the Heat Equation, Publish or Perish (1974).
13. L.S.Brown and D.B.Creamer, Phys. Rev. D18 (1978) 3695.
14. E.Corrigan, P.Goddard and S.Templeton, Nucl. Phys. B151 (1979) 93.
15. V.G.Drinfeld and Yu.I.Manin, Yad. Fiz. 29 (1979) 1646.
16. A.A.Belavin, V.A.Fateev, A.S.Schwarz and Yu.S.Tyupkin, Phys. Letts. 83B (1979) 317.

17. B.Berg and M.Lüscher, Nucl. Phys. B160 (1979) 281.

18. I.Jack, Cambridge preprint DAMTP 79/21.

19. I.V.Frolov and A.S.Schwarz, JETP Letts. 28 (1978) 249.
V.A.Fateev, I.V.Frolov and A.S.Schwarz, Nucl. Phys. B154 (1979) 1.
B.Berg and M.Lüscher, Comm. Math. Phys. 69 (1979) 57.

20. But see ref 16 for an idea on this.