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A NOTE ON MEASURABILITY OF TRAJECTORIES OF A STOCHASTIC PROCESS

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In this note, we present an alternative method of studying the measurability properties of trajectories of a stochastic process. More precisely, we will consider the following situation: Given any vector or scalar measure

$$\vec{\mu}: \mathcal{B}(X^{\langle 0,1\rangle}) \to E$$

on a family $\mathcal{B}(X^{\langle 0,1\rangle})$ of all Baire subsets of $X^{\langle 0,1\rangle}$, where X is some locally compact metrizable space X, consider the question: On which trajectories lives $\vec{\mu}$?

Our method is based on the identification of each Baire function $f \in X^{\langle 0,1\rangle}$ (or, rather, the corresponding a.e. equivalence class) with the probability α_f on $\langle 0,1\rangle \times X$, determined uniquely by the requirements:

i) α_f is carried by a graph of f

ii) the projection $\pi(\alpha_f)$ of α_f on $\langle 0,1\rangle$ is just the Lebesgue measure .

Thus, by identifying f with α_f, the topology of convergence in measure on functions can be induced by the weak* topology on the space $\mathcal{P}(\langle 0,1\rangle \times \check{X})$ of all Radon probabilities on $\langle 0,1\rangle \times \check{X}$ (where $\check{X}$ denotes a one point compactification of X).

Now we generalize the notion of a trajectory.

Denote by

$$\mathcal{T} = \{\alpha \in \mathcal{P} , \pi_{\langle 0,1\rangle}(\alpha) = \text{Lebesgue measure}\}$$ "generalized trajectories"

$\mathcal{T}_{meas} = \{ \alpha \in \mathcal{T},\ \alpha$ is carried by a graph of some Baire function $f \in X^{\langle 0,1 \rangle} \}$ "measurable trajectories"

$\mathcal{T}_{cont} = \{ \alpha \in \mathcal{T},\ \alpha$ is carried by a graph of some a.e. continuous function $f \in X^{\langle 0,1 \rangle} \}$ "a.e. continuous trajectories" .

Now, we "carry on" the measure $\vec{\mu} : \mathcal{B}(X^{\langle 0,1 \rangle}) \to E$ in some very natural way to the space $\mathcal{T}$.

Then we will investigate the support of the resulting measure. We will show, at the end of this note, that each measure on $\mathcal{T}_{meas}$ can be viewed as a measure on $X^{\langle 0,1 \rangle}$.

Our results will then be comparable to the classical ones (see e.g. [1 , Th.III,3,1]). Actually, they generalize them slightly.

<u>Definition</u>

Denote by $P_{IA}(\alpha) = \alpha(I \times A)$

($P_{IA}(\alpha)$ says, "how often α dwells in A during I) .

Suppose that each map

$$\{ t_1 \dots t_n \to \vec{\mu}(\pi^{-1}_{t_1 \dots t_n}(A_1 \times \dots \times A_n)) \} : \langle 0,1 \rangle^n \to E$$

is Lusin measurable, whenever A_i are open in X .

Put $$\tilde{\mu}(P_{I_1 A_1} \cdot \ldots \cdot P_{I_n A_n}) = \int_{I_1 \times \dots \times I_n} \vec{\mu}(\pi^{-1}_{t_1 \dots t_n}(A_1 \times \dots \times A_n)) . dt_1 \dots dt_n ,$$

where $\pi_{t_1 \dots t_n} : X^{\langle 0,1 \rangle} \to X^{\{t_1 \dots t_n\}}$ denotes the canonical projection.

<u>Theorem 1.</u> $\tilde{\mu}$ extends uniquely to a vector measure

$\tilde{\mu} : \mathcal{B}(\mathcal{T}) \to E$.

<u>Example.</u> It is not true in general, that $\tilde{\mu}$ lives on $\mathcal{T}_{meas}$!

For an arbitrary Baire probability on X , put

$\mu(\pi^{-1}_{t_1 \dots t_n}(A_1 \times \dots \times A_n) = \prod_1^n \nu(A_i)$ and extend this according to the Kolmogorov theorem.

The resulting probability on $X^{\langle 0,1 \rangle}$ gives rise to a measure $\tilde{\mu}$, which is, as can be easily shown, supported by a single element of $\mathcal{T}$, namely by $\lambda \otimes \nu$, where λ denotes the Lebesgue measure!

Notation. Denote by

$$I_\delta = \{(t,s) \in \langle 0,1 \rangle \, | \, |t-s| < \delta\}$$
$$\lambda_\delta = \frac{1}{2\delta} \lambda$$
$$I_\delta^n = \{(t_1 \dots t_n) \in \langle 0,1 \rangle^n \, , \, |t_i - t_j| < \delta\}$$
$$\lambda_\delta^n = \left(\frac{1}{2\delta}\right)^{n-1} \lambda \quad .$$

Theorem 2. $\tilde{\mu}$ has the support in $\mathcal{T}_{meas} \Longleftrightarrow$ for each $\varepsilon > 0$ and A open the following holds:

$$\delta \searrow 0 \Longrightarrow \lambda_\delta \{(t,s), \| \vec{\mu}(\pi_t^{-1}(A) \triangle \pi_s^{-1}(A) \| > \varepsilon \} \longrightarrow 0 \, .$$

Example. The latter condition holds e.g. in the case, when $\| \vec{\mu}(\pi_s^{-1}(A) \triangle \pi_t^{-1}(A) \| \xrightarrow[s \to t]{} 0$ holds for almost all $t \in \langle 0,1 \rangle$. This corresponds to the a.e. stochastic continuity of the process.

Theorem 3. $\tilde{\mu}$ has the support in $\mathcal{T}_{cont} \Longleftrightarrow$ for each $\varepsilon > 0$ and A open the following holds:

$$\delta \searrow 0 \Longrightarrow \lambda_\delta^n \{(t_1 \dots t_n), \| \pi_{t_i}^{-1}(A) \setminus \pi^{-1}_{t_1 \dots t_n}(A \times \dots \times A) \| > \varepsilon \text{ holds for some } i\} \longrightarrow 0 \text{ uniformly with respect to } n \in N \, .$$

Note. For Markov processes, this gives rather weak results (a.e. continuity instead of the nonexistence of the discontinuities of 2. kind).

In order to prove an analogy of Th. 3 for trajectories without discontinuities of 2. kind, we should use some more subtle Markovian arguments.

Finally, let us show that if $\tilde{\mu}$ has the support in $\mathcal{T}_{meas}$, then it can be viewed as a measure on $X^{\langle 0,1\rangle}$. Choose a partition of $\langle 0,1\rangle$ consisting of all intervals

$$I_n^i = \langle \frac{i}{2^n}, \frac{i+1}{2^n} \rangle \quad , i=0,1,\dots,2^n-1 .$$

For each $t\in\langle 0,1\rangle$ choose $i(n)$ such that $t\in\langle \frac{i}{2^n}, \frac{i+1}{2^n})$.

Call $t\in\langle 0,1\rangle$ a Lebesgue point of α_f if there exists $y\in X$ such that for each A open containing y,

$$\lim_{n\to\infty} (2^n P_{I_{i(n)}^A}(\alpha)) = 1 .$$

Then it can be shown, that there exists a null set $N\subset\langle 0,1\rangle$ such that the following is true:

For each $t\in\langle 0,1\rangle\setminus N$, t is the Lebesgue point of $\tilde{\mu}$ almost all trajectories and if we fix some $x_o\in X$ and put

$\mathcal{F}(t) = y$ whenever t is a Lebesgue point of α_f,

$\mathcal{F}(t) = X_o$ otherwise

then the map

$\{\alpha_f \rightsquigarrow \mathcal{F}\} : \mathcal{T}_{meas} \to X^{\langle 0,1\rangle}$ is Baire measurable and the image of $\tilde{\mu}$ coincides with μ on $\mathcal{B}(X^{\langle 0,1\rangle\setminus N})$.

References

[1] Gihman-Skorochod: The theory of stochastic processes. Part I .