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Scattered compactifications

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SCATTERED COMPACTIFICATIONS

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A space is said to be scattered if it does not contain a dense in itself subset. E.g., the space $W(\alpha) = \{\xi : \xi < \alpha\}$ is scattered in the open interval topology for any ordinal α.

Z.Semadeni [10] posed the following question: Is there a scattered compactification for any completely regular scattered space ? In particular: Is there a scattered compactification for $N \cup \{p\} \subset \beta N$ where $p \in \beta N - N$?

In this contribution we present a survey of results concerning the above questions.

(1) [4] Each metrizable separable scattered space can be embedded in $W(\alpha)$ for some $\alpha < \omega_1$ and thus it admits a scattered compactification.

Example 3.6, p. 17, in [10] is incorrect. We have

(2) [14] Each metrizable scattered space can be embedded in $W(\alpha)$ for some ordinal α, and thus it admits a scattered compactification

Similarly as in [14] one can prove the following.

(3) Each suborderable paracompact scattered space can be embedded in $W(\alpha)$ for some ordinal α, and thus it admits a scattered compactification.

(4) [7] Each suborderable scattered space of countable height is orderable and admits an orderable scattered compactification.

(5) [14] If X is a paracompact scattered space, then ind X = = Ind X = dim X = 0.

Hence we have

(6) If X admits a scattered compactification, then ind X = 0.

It is easy to prove that

(7) If βX contains a non-degenerate continuum C such that $C \cap X \neq 0$, then X has no scattered compactification.

(8) [8] (CH) Example of a completely regular scattered space X of cardinality $\aleph_1$ with ind X > 0; by (6), X has no scattered compactification.

(9) [12] Example of a completely regular scattered space X of the cardinality $2^{\aleph_0}$ with ind X>0; by (6), X has no scattered compactification.

(10) [9] (CH) If p is a P-point in $\beta N - N$, then $N \cup \{p\}$ admits a scattered compactification.

A point $p \in \beta N - N$ is said to be a P-point of order 2 in $\beta N - N$ if there exists a countable set M of P-points in $\beta N - N$ such that p is a P-point in $\overline{M} - M$.

(11) [2] (CH) If p is a P-point of order 2 in $\beta N - N$, then $N \cup \{p\}$ admits a scattered compactification.

(12) [3] (CH) If p is a P-point of order α in $\beta N - N$, where $\alpha < \omega_1$, then $N \cup \{p\}$ admits a scattered compactification.

(13) [11] If p belongs to the carrier of a regular Borel non-atomic measure μ on βN with $\mu(\beta N) = 1$, then $N \cup \{p\}$ has no scattered compactification.

Let us note that the carrier of a regular Borel non-atomic measure μ on βN with $\mu(\beta N) = 1$ is a perfect subset of $\beta N - N$ satisfying the countable chain condition.

(14) [15] If p belongs to a perfect subset of $\beta N - N$ satisfying the countable chain condition, then $N \cup \{p\}$ has no scattered compactification.

(15) [15] The set $\{p \in \beta N - N : N \cup \{p\}$ has no scattered compactification$\}$ is $\aleph_0$-bounded; hence, in particular, it is countably compact.

A point $p \in S$ is said to be a point of extremal disconnectedness of S if given two disjoint open sets U and V in S we have $p \notin \overline{U} \cap \overline{V}$. Let S^{ed} denote the set of all points of extremal disconnectedness of S. Note that each subset of $\beta N - N$ satisfying the countable chain condition is extremally disconnected.

(16) [16] If βX contains a dense in itself subset S such that $X \cap S^{ed} \neq 0$, then X has no scattered compactification.

If X is not pseudocompact, then βX contains a copy of βN and thus βX cannot be scattered. Moreover

(17) [5] If X is not pseudocompact, then there exists a point $p \in \beta X - X$ such that $X \cup \{p\}$ has no scattered compactification.

(18) [6] Let X be a product of uncountably many two-point discrete spaces. Let one point be given a base of neighborhoods as in the product topology; let all other points of X be isolated. Then X has no scattered compactification.

A filter F is called κ-regular if it contains a subset F_0 of cardinality κ such that the intersection of infinitely many members of F_0 is empty.

(19) [13] If X is a space with the only non-isolated point p, card $X > \aleph_1$ and the neighborhood base at p restricted to $X - \{p\}$ is κ-regular, where $\aleph_1 < \kappa \leq$ card X, then X has no scattered compactification.

(20) [1] Example of a countable space with one non-isolated point all compactifications of which contain βN.

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