Jan Menu

Reflective subcategories of Poset and Top

In: Zdeněk Frolík (ed.): Abstracta. 5th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1977. pp. 51--60.

Persistent URL: http://dml.cz/dmlcz/701089

FIFTH WINTER SCHOOL (1977)

REFLECTIVE SUBCATEGORIES OF *Poset* AND *Top*

Jan MENU

1. INTRODUCTION.

1.1. In this paper reflective subcategories of some concrete categories are studied. It is proved that *Poset* has only one non-trivial reflective subcategory. This result is used to describe the type of reflection in *Top* by means of the separation axioms that are satisfied.

1.2. Let C be a concrete category, i.e. a couple (C,U) where C is a category and $U : C \to Set$ a faithful functor. For a set X we denote

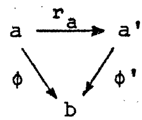
$$C \cup X$$

the class of all $a \in \text{obj } C$ such that $U(a) = X$. If $a,b \in C \cup X$, we define

$$a < b \text{ iff there exists a } \phi : a \to b \in C, \text{ with } U(\phi) = 1_X$$

and this defines a preorder on $C \cup X$.

1.3. A subcategory K of C is said to be reflective iff for every $a \in \text{obj } C$, there exists an $a' \in \text{obj } K$ and a morphism $r_a : a \to a'$ such that for every morphism $\phi : a \to b$, $b \in \text{obj } K$ there exists a unique $\phi' : a' \to b$ which makes the diagram



commute.

K is said to be epi- (resp. mono-) reflective iff the reflection morphisms r_a are epi- (resp. mono-) morphisms. A subcategory of concrete category is simply reflective iff the reflection morphisms are carried by the identity.

It is well-known that every mono-reflective subcategory is also epi-reflective.

Evidently C is a reflective subcategory of itself, and if C has a terminal object t_C, the subcategory consisting of this single object is also reflective. In these cases the reflective subcategory is said to be trivial.

1.4. In the category of topological spaces the subcategories Top_0, Top_1, Top_2, CR_1, CR_2 (with as objects the (T_0)-, (T_1)-, (T_2)-completely regular, completely regular Hausdorff-spaces) are examples of epi-reflective subcategories, while the subcategory of the compact Hausdorff-spaces is an example of a non-epi-reflective subcategory.

1.5. Let (C,U) be a concrete category, $2 = \{1,2\}$ the 2-element set, $p \in C \cup 2$. If $a \in \text{obj } C$, $x \in U(a)$, then define

$$\begin{aligned} C'_p(x) &= \{y \mid \exists\, \phi : p \to a \in C,\ U(\phi)(2) = \{x,y\}\} \\ C^0_p(x) &= \{x\} \\ C^k_p(x) &= C'_p(C^{k-1}_p(x)) \\ C_p(x) &= \cup\{C^k_p(x) \mid k \in \mathbb{N}\}. \end{aligned}$$

a is said to be p-connected (p-c) iff

$$\forall\, x \in U(a),\ C_p(x) = U(a).$$

a is p-totally disconnected (p-td) iff

$$\forall x \in U(a) : C_p(x) = \{x\}.$$

It is easy to see that if C is complete and has extremal subobjects the full subcategory of the p-td objects is epi-reflective.

1.6. In the category *Poset* of partially ordered sets and monotone mappings we denote by

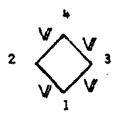
1 the poset · 1

2 " " · 1 · 2

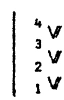
2' " "

2" " "

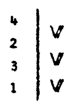
C " "



D " "



E " "



§2. REFLECTIVE SUBCATEGORIES OF *Poset*.

2.1. PROPOSITION 1. Let K be a non-trivial reflective subcategory of *Poset*, $K \neq K_1$. Then K is simply reflective.

PROOF. Let $r_2' : 2' \to \overline{2'}$ be the reflection of 2'.

a) Suppose $r_{2'}(2') = 1$. Because every constant function in *Poset* carries a morphism, and because of the uniqueness condition in the definition of reflection, it follows that $\overline{2'} = 1$.

Let $(X,<)$ be a 2'-connected poset, $x \in X$ and $y \in C'_{2'}(x)$. We denote by

$$f_{x,y} : 2' \to (X,<)$$

the morphism such that $f_{x,y}(2') = \{x,y\}$.

Let $(X,<) \xrightarrow{r_X} (X_K,<_K)$ be the reflection of $(X,<)$, then there exists a unique $f'_{x,y} : 1 \to (X_K,<_K)$ such that the diagram

$$\begin{array}{ccc} (X,<) & \xrightarrow{r_X} & (X_K,<_K) \\ {\scriptstyle f_{x,y}}\uparrow & & \uparrow{\scriptstyle f'_{x,y}} \\ 2' & \xrightarrow{r'_2} & 1 \end{array}$$

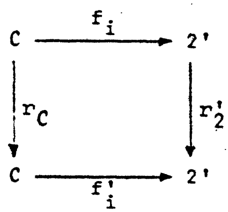
commutes. Consequently, $r_X(X) = 1$. Again it follows easily that $(X_K,<_K) = 1$ for every 2'-connected $(X,<)$.

Let $r_2 : 2 \to \overline{2}$ be the reflection of 2.

(i) If $r_2(2) = 1$, and thus $\overline{2} = 1$, one proves with the same method as before that $\forall\ (X,<) \in$ obj *Poset* : $(X_K,<_K) = 1$, and in this case the reflection is trivial

(ii) Suppose that r_2 is one-to-one. Because $\overline{2}$ is necessarily 2'-td, it follows that r_2 is an isomorphism and $K = K_1$.

b) If r'_2 is one-to-one, then also r_2 is one-to-one. Let $r_C : C \to \overline{C}$ be the reflection of C. Consider the diagram



with $\quad f_1(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } 3 \\ 2 & \text{if } x = 2 \text{ or } 4 \end{cases}$

$\quad f_2(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } 2 \\ 2 & \text{if } x = 3 \text{ or } 4 \end{cases}$

and f'_i $(i = 1,2)$ the unique extension of f_i which makes the diagram commute.

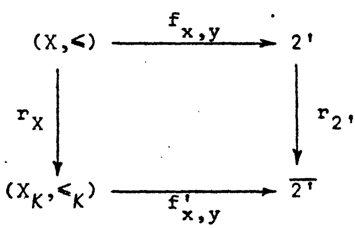
From this it follows that r_C is one-to-one.

Let $(X,\leq) \in$ obj *Poset*, $x \neq y \in X$.

(i) if $x < y$, consider the morphism

$f_{x,y} : (X,\leq) \to 2' : \begin{cases} z \to 2 & \text{if } z \geq y \\ z \to 1 & \text{for the others.} \end{cases}$

Because the diagram



commutes, it follows that $r_X(x) \neq r_X(y)$.

(ii) if $x \nleq y$ and $y \nleq x$, consider the morphism

$g_{x,y} : (X,\leq) \to C : \quad \begin{array}{l} z \to 1 \\ z \to 3 \\ z \to 4 \\ z \to 2 \end{array} \left\{ \begin{array}{l} \text{if } z < x \quad \text{or} \quad z < y \\ \text{if } z = x \\ \text{if } z > x \quad \text{or} \quad z > y \\ \text{for the others.} \end{array} \right.$

As before, because the diagram

$$\begin{array}{ccc}
(X,<) & \xrightarrow{g_{x,y}} & C \\
\downarrow{\scriptstyle r_X} & & \downarrow{\scriptstyle r_C} \\
(X_K,<_K) & \xrightarrow[g'_{x,y}]{} & \overline{C}
\end{array}$$

commutes, it follows that $r_C(x) \neq r_C(y)$, and thus r_X is one-to-one for every $(X,<) \in$ obj *Poset*. K is then monoreflective, and 1.3. states that K is epi-reflective, and thus simply reflective.

2.2. PROPOSITION 2. Let K be a simply reflective subcategory of *Poset*. Then K = *Poset*.

PROOF. Because D and E are maximal in *Poset* $\cup$ 4, D and E $\in$ obj K, and thus $C \in$ obj K.

Let $(X,<) \in$ obj *Poset*, $x \neq y \in X$, $x \nless y$ and $y \nless x$. It follows from the diagram

$$\begin{array}{ccc}
(X,<) & \xrightarrow{1_X} & (X_K,<_K) \\
\downarrow{\scriptstyle g_{x,y}} & & \downarrow{\scriptstyle g'_{x,y}} \\
C & \xrightarrow[1_C]{} & C
\end{array}$$

that in $(X_K,<_K)$: $x \nless y$ and $y \nless x$, which proves that K = *Poset*.

2.3. COROLLARY 1. K_1 is the only non-trivial reflective subcategory of *Poset*.

§3. REFLECTIVE SUBCATEGORIES OF *Top*.

3.1. PROPOSITION 3. (*Herrlich*) Let K be a reflective subcategory of *Top*. Consider L, the full subcategory generated

by the subspaces of products of objects in K.
Then the following hold :

(i) L is epireflective in *Top*

(ii) K is epireflective in L and the reflection morphisms are embeddings

(iii) $K = L$ iff K is epireflective.

3.2. Most natural examples of reflective subcategories are epireflective or simply reflective. They were characterised in [2] and [3].

PROPOSITION 4. A subcategory K of *Top* is epireflective iff every product of objects in K and subspaces of objects in K are again in K.

PROPOSITION 5. A reflective subcategory K of *Top* is simply reflective iff it is epireflective and every indiscrete space is in K.

3.3. The following proposition gives a characterisation of epireflective subcategories of *Top* by means of separation-axioms :

PROPOSITION 6. Let K be an epireflective subcategory of *Top*. Then one of the following holds :

(i) K is simply reflective

(ii) $K = Top_0$

(iii) $K \subset Top_1$.

PROOF. (i) Suppose a non (T_0)-space $(X,T) \in \text{obj}\, K$. Then (X,T) contains the indiscrete space on the 2-element set as a subspace. Because every indiscrete space is a subspace of a product of the

2-element indiscrete space, Propositions 4 and 5 prove that K is simply reflective.

(ii) Let $K \subset Top_0$, then K is epireflective in Top_0. Given (X,T) a (T_0)-space, x and y $\in$ X, define a partial ordering on X as follows :

$$x < y \leftrightarrow x \in \overline{y}$$

and this we denote by $(p(X,T),<_T)$.

Define

$$F : Top_0 \to Poset$$

as follows :

$$F((X,T)) = (p(X,T),<_T)$$

$$F((T,f,T')) = (<_T,f,<_{T'}).$$

It is easy to see that F is a functor.

We now prove that $K' = F(K)$ is a reflective subcategory of Poset.
If $(X,<) \in$ obj Poset, define

$$T = \cup\{U \mid (p(X,U),<_U) = (X,<)\}$$

$r : (X,T) \to (X',T')$ the reflection morphism

$$(X',<') = p((X',T'),<_{T'}).$$

Let $f : (X,<) \to (Y,\subset)$ be a monotone function, $(Y,\subset) \in$ obj K'. Then there exists a $(Y,U) \in$ obj K such that $p((Y,U),<_U) = (Y,\subset)$. The function $f : (X,T) \to (Y,U)$ is continuous, and there exists a unique $f' : (X',T') \to (Y,U)$ such that $f' \circ r = f$. Because also $(<',f',\subset)$ is a morphism, K' is reflective in Poset. It now follows from Corollary 2.3. that K' is trivial or the category of 2-td objects. In the last case, $K \subset Top_1$, which is also true if $K' = \{1\}$.

Suppose now that $K' =$ Poset. If $U = \{\{1\},\{1,2\}\}$, then $(2,U) \in$ obj K.

Let $(X,\mathcal{T}) \in$ obj $\mathcal{T}op_0$, $x \in X$, $V \in \mathcal{V}(x)$, V open. Define

$$f_{x,V} : X \to z : z \mapsto 1 \quad \text{if} \quad z \in V$$
$$z \mapsto 2 \quad \text{if} \quad z \notin V.$$

$f_{x,V} : (X,\mathcal{T}) \to (2,\mathcal{U})$ is continuous, which proves that $r : (X,\mathcal{T}) \to (X',\mathcal{T}')$ is an isomorphism and $\mathcal{K} = \mathcal{T}op_0$.

3.4. The following example shows that a non-epireflective subcategory of $\mathcal{T}$ need not consist of only Hausdorff-spaces.

Let

$$\mathcal{D} = \{n | n \in \mathbb{N}\} \cup \{\alpha,\beta\}, \ \mathcal{U} = \{A | A \subset D,\ A \cap \{\alpha,\beta\} = \phi \text{ or } A^c \text{ finite}\}$$

and $(X,\mathcal{T})$ is the topological sum of $(\mathbb{R},\mathcal{T}_E)$ and $(\mathcal{D},\mathcal{U})$. Let K be the full subcategory of $\mathcal{T}op$ with as objects the spaces $(A,\mathcal{V})$, subspace of a product of $(X,\mathcal{T})$, and with the following property : whenever $B \subset A$, B connected $\Rightarrow$ $\overline{B} \subset A$. It is easy to see that $\mathcal{K}$ is a reflective subcategory of $\mathcal{T}op$, but not epireflective.

REFERENCES

[1] BIRKHOFF G., Lattice Theory,
American Math. Soc., Colloq. Publ., Vol. 25, 1940.

[2] HERRLICH H., Topologische reflektionen und Coreflektionen.
Lecture Notes in Mathematics 78, Springer Verlag,
Berlin-Heidelberg-New York.

[3] KENNISON J.F., Reflective functors in General Topology and elsewhere.
Trans. A.M.S. 118 (1965), 303-315.

[4] MacLANE S., Categories. For the working mathematician.
Graduate Texts in Mathematics, vol. 5.

[5] MENU J., Concrete categories.
Dissertation (in Dutch).