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## A COMMON GENERALIZATION OF KELLEY'S THEOREM (CONCERNING MEASURES ON A BOOLEAN ALGEBRA) AND OF VON NEUMANN'S

## MINIMAX THEOREM

by

Marek WILHELM

Let  $(X, +)$  be a commutative semigroup with 0, let  $\omega : X \rightarrow [0, \infty]$  be subadditive ( $\omega(x+y) \leq \omega(x) + \omega(y)$ ) and monotone ( $\omega(x) \leq \omega(x+y)$ ), and let  $A$  denote the family of all additive functionals  $\xi : X \rightarrow [0, \infty]$  ( $\xi(x+y) = \xi(x) + \xi(y)$ ). For every subset  $Y \subset X$  we define the number

$$K_{\omega}(Y) = \inf \left\{ \frac{1}{k} \omega \left( \sum_{i=1}^k y_i \right) : y_1, \dots, y_k \in Y \right\},$$

where  $y_i$  are not necessarily distinct; with the convention that  $\inf \emptyset = \infty$ .

Theorem. We have

$$K_{\omega}(Y) = \sup \left\{ \inf_{y \in Y} \xi(y) : A \ni \xi \leq \omega \right\},$$

the supremum being attained.

Corollary 1. (i)  $\iff$  (ii),

(i) there exists  $\xi \in A$  with  $\xi \leq \omega$  and  $\xi(y) > 0$  for all  $y \in Y$ ,

(ii) there are  $Y_n \subset X$  with  $Y \subset \bigcup_{n=1}^{\infty} Y_n$  and  $K_{\omega}(Y_n) > 0$  for all  $n = 1, 2, \dots$

Corollary 2. Let  $(X, +, \cdot)$  be a semilinear space, and let  $\omega$  be, moreover, positively homogeneous (i.e.  $\omega(\alpha x) = \alpha \omega(x)$  for  $x \in X$ ,  $\alpha \in \mathbb{R}^+$ ). Then

$$\begin{aligned} K_{\omega}(Y) &= \inf \{ \omega(y) : y \in \text{conv } Y \} = \\ &= \sup \{ \inf_{y \in Y} \xi(y) : L \ni \xi \leq \omega \} , \end{aligned}$$

the supremum being attained.

Here  $L$  denotes the family of all  $\xi \in A$  that are positively homogeneous (semilinear).

Corollary 3 (J.L. Kelley). Let  $\mathcal{A}$  be a Boolean algebra, and for every non-empty subset  $\mathcal{B} \subset \mathcal{A}$  define the number

$$I(\mathcal{B}) = \inf \left\{ \frac{i(\xi)}{n(\xi)} \right\},$$

where  $\xi$  runs over all

finite collections of elements of  $\mathcal{B}$  (not necessarily distinct),  $n(\xi)$  is the number of elements of  $\xi$ , and  $i(\xi)$  is the maximal number of elements of  $\xi$  with non-empty intersection. Then

$$I(\mathcal{B}) = \sup \left\{ \inf_{B \in \mathcal{B}} m(B) \right\},$$

the supremum, taken over all finite-

additive measures  $m: \mathcal{A} \rightarrow [0, 1]$ , being attained.

There exists a strictly positive ( $m(A) > 0$  for  $A \neq \emptyset$ ) finite-additive measure  $m$  on  $\mathcal{A}$  iff there are  $\mathcal{B}_n \subset \mathcal{A}$  with  $\mathcal{A} \setminus \{\emptyset\} \subset \bigcup_{n=1}^{\infty} \mathcal{B}_n$  and  $I(\mathcal{B}_n) > 0$  for all  $n = 1, 2, \dots$

Corollary 4 (J.von Neumann). Let  $S$  and  $T$  be arbitrary non-empty finite sets. Then for every function  $h: S \times T \rightarrow (-\infty, \infty]$  we have

$$\inf \left\{ \max_{s \in S} \sum_{t \in T} f(t)h(s,t) : f: T \rightarrow [0,1], \sum_{t \in T} f(t) = 1 \right\} =$$

$$= \sup \left\{ \min_{t \in T} \sum_{s \in S} g(s)h(s,t) : g: S \rightarrow [0,1], \sum_{s \in S} g(s) = 1 \right\},$$

the supremum and infimum being attained.