Marek Wilhelm

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In: Zdeněk Frolík (ed.): Abstracta. 4th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1976. pp. 155--157.

Persistent URL: http://dml.cz/dmlcz/701067

FOURTH WINTER SCHOOL (1976)

A COMMON GENERALIZATION OF KELLEY'S THEOREM (CONCERNING MEASURES ON A BOOLEAN ALGEBRA) AND OF VON NEUMANN'S MINIMAX THEOREM

by

Marek WILHELM

Let $(X,+)$ be a commutative semigroup with 0, let $\omega : X \longrightarrow [0,\infty]$ be subadditive $(\omega(x+y) \leq \omega(x) + \omega(y))$ and monotone $(\omega(x) \leq \omega(x+y))$, and let A denote the family of all additive functionals $\xi : X \longrightarrow [0,\infty]$ $(\xi(x+y) = \xi(x) + \xi(y))$. For every subset $Y \subset X$ we define the number

$K_\omega(Y) = \inf\{\frac{1}{k}\,\omega(\sum_{i=1}^{k} y_i) : y_1,\dots,y_k \in Y\}$, where y_i are not necessarily distinct; with the convention that $\inf \emptyset = \infty$.

Theorem. We have

$K_\omega(Y) = \sup\{\inf_{y\in Y} \xi(y) : A \ni \xi \leq \omega\}$, the supremum being attained.

Corollary 1. (i)$\Longleftrightarrow$(ii),

(i) there exists $\xi \in A$ with $\xi \leq \omega$ and $\xi(y) > 0$ for all $y \in Y$,

(ii) there are $Y_n \subset X$ with $Y \subset \bigcup_{n=1}^{\infty} Y_n$ and $K_\omega(Y_n) > 0$ for all $n = 1,2,\dots$

Corollary 2. Let $(X,+,\cdot)$ be a semilinear space, and let ω be, moreover, positively homogeneous (i.e. $\omega(\alpha x) = \alpha\omega(x)$ for $x \in X$, $\alpha \in R^+$). Then

$$K_\omega(Y) = \inf\{\omega(y): y \in \text{conv } Y\} =$$
$$= \sup\{\inf_{y \in Y} \xi(y): L \ni \xi \leq \omega\}$$, the supremum being attained.

Here L denotes the family of all $\xi \in A$ that are positively homogeneous (semilinear).

Corollary 3 (J.L. Kelley). Let $\mathcal{A}$ be a Boolean algebra, and for every non-empty subset $\mathcal{B} \subset \mathcal{A}$ define the number

$$I(\mathcal{B}) = \inf\left\{\frac{i(\xi)}{n(\xi)}\right\},$$ where ξ runs over all

finite collections of elements of $\mathcal{B}$ (not necessarily distinct), $n(\xi)$ is the number of elements of ξ, and $i(\xi)$ is the maximal number of elements of ξ with non-empty intersection. Then

$I(\mathcal{B}) = \sup\{\inf_{B \in \mathcal{B}} m(B)\}$, the supremum, taken over all finitely additive measures $m: \mathcal{A} \to [0,1]$, being attained.

There exists a strictly positive ($m(A) > 0$ for $A \neq 0$) finitely additive measure m on $\mathcal{A}$ iff there are $\mathcal{B}_n \subset \mathcal{A}$ with $\mathcal{A} \setminus \{0\} \subset \bigcup_{n=1}^{\infty} \mathcal{B}$ and $I(\mathcal{B}_n) > 0$ for all $n = 1,2,\ldots$

Corollary 4 (J.von Neumann). Let S and T be arbitrary non-empty finite sets. Then for every function $h: S \times T \to (-\infty, \infty]$ we have

$$\inf\left\{\max_{s\in S}\sum_{t\in T} f(t)h(s,t):\ f\colon T\longrightarrow[0,1],\ \sum_{t\in T} f(t)=1\right\}=$$

$$=\sup\left\{\min_{t\in T}\sum_{s\in S} g(s)h(s,t):\ g\colon S\longrightarrow[0,1],\ \sum_{s\in S} g(s)=1\right\},$$

the supremum and infimum being attained.