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SOME THEOREMS ON MEASURABLE AND CONTINUOUS SELECTIONS AND SOME APPLICATIONS

by

G. MÄGERL

Let X, Y be sets ($\neq \emptyset$), $\mathcal{O}\!\mathcal{L} \subseteq \mathcal{P}(X)$, $\mathcal{B} \subseteq \mathcal{P}(Y)$. Call $\Phi : X \longrightarrow Y$ a correspondence, iff $\Phi(x)$ is a nonempty subset of Y, $\forall x$. Call a map $f: X \longrightarrow Y$ (correspondence $\Phi : X \longrightarrow Y$) $\mathcal{O}\!\mathcal{L}$-$\mathcal{B}$-measurable, iff $f^{-1}(B) \in \mathcal{O}\!\mathcal{L}$ ($\Phi^{-1}(B) = \{x \mid \Phi(x) \cap B \neq \emptyset\} \in \mathcal{O}\!\mathcal{L}$) $\forall B \in \mathcal{B}$. $f: X \longrightarrow Y$ is a selection of Φ, iff $f(x) \in \Phi(x)$ $\forall x$.

Definition. Y topological space, $\sigma : \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$ a map such that $\sigma(Y) = Y$. Then $E(X,Y,\sigma) \Longleftrightarrow \exists\, Z \subseteq Y$ dense, $\forall$ coverings $\{A_z \mid z \in Z\}$ of $X (A_z \in \mathcal{O}\!\mathcal{L})$ $\exists f: X \longrightarrow Y$ $\mathcal{O}\!\mathcal{L}$-$\mathcal{U}$-measurable ($\mathcal{U}$ = open sets) such that $f(x) \in \sigma\{z \mid x \in A_z\}$ $\forall x \in X$.

Examples are:

$(X, \mathcal{O}\!\mathcal{L})$ measurable space, Y Polish, σ = id, then $E(X,Y,\sigma)$.
X paracompact, $\mathcal{O}\!\mathcal{L} = \mathcal{U}$, Y locally convex space, $\sigma(T) = \text{conv}(T)$ $(T \subseteq Y)$ then $E(X,Y,\sigma)$.

From this one gets a simultaneous proof of Theorems of KURATOWSKI/RYLL-NARDZEWSKI on measurable selections and MICHAEL on continuous selections, namely.

Suppose $\mathcal{O}\!\mathcal{L}$ $\cap^{\text{finite}}$, $\cup^{\text{countable}}$-stable, Y complete metric, (V_n) a fundamental sequence of entourages for Y such that $\sigma(V_n(y)) \subseteq V_n(y)$ $\forall m$ $\forall y$, $V_n \circ V_n \subseteq V_{n-1}$, V_n symmetric, $V_n(y)$ open. Suppose $E(X,Y,\sigma)$ is true. Let Φ : $: X \longrightarrow Y$ be an $\mathfrak{A}-\mathfrak{V}$-measurable correspondence such that $\forall x$ $\forall n$ $\sigma(V_n(\Phi(x))) \subseteq V_n(\Phi(x))$. Then $\exists f: X \longrightarrow Y$ $\mathfrak{A}-\mathfrak{V}$-measurable, such that $f(x) \in \overline{\Phi(x)}$ $\forall x$.

As a consequence we get:

Theorem (KURATOWSKI/RYLL-NARDZEWSKI).
$(X,\mathfrak{A})$ measurable space, Y Polish, $\Phi : X \longrightarrow Y$ $\mathfrak{A}-\mathfrak{V}$-measurable (or $\mathfrak{A}-\mathfrak{F}$-measurable, $\mathfrak{F}$ = closed sets) with closed values, then Φ has an $\mathfrak{A}-\mathfrak{B}(Y)$-measurable selection ($\mathfrak{B}(Y)$ = Borel subsets of Y).

Corollary. X, Y topological spaces, $\Phi : X \longrightarrow Y$, such that $G(\Phi) = \{(x,y) \mid y \in \Phi(x)\}$ is the Hausdorff continuous image of a Polish space, μ a Borel measure on X, then Φ has a $\mathfrak{B}(X)^* - \mathfrak{B}(Y)$-measurable selection ($\mathfrak{B}(X)^*$ = = Caratheodory completion of $\mathfrak{B}(X)$); iff X is locally compact and μ a Radon measure, then one can replace $\mathfrak{B}(X)^*$ by $\mathfrak{M}$, the μ-measurable sets.

Applications: Implicit function theorems (FILIPPOV's Lemma) and BANG-BANG-principles in control theory, integration of correspondences. Extensions of measures and preimages of measures.

Theorem (YERSHOV, 1970, LUBIN, 1974, LANDERS/ROGGE, 1974).

(1) Suppose X Souslin space, $\mathfrak{B} \subseteq \mathfrak{B}(X)$ countably generated. Then every measure on $\mathfrak{B}$ has an extension to $\mathfrak{B}(X)$.

(2) X, Y Souslin spaces, $f: X \longrightarrow Y$ onto, Borel map, μ Borel measure on Y, then $\exists$ Borel measure ν on X, such that $f(\nu) = \mu$.

Proposition (proved for compact metric spaces and continuous f by EISELE, 1975).

X Luzin, Y Souslin, μ Borel measure on Y, $f: X \longrightarrow Y$ onto, Borel, if the preimage of μ under f is unique, then $\mu(\{y \mid \text{card } f^{-1}(y) \geqq 2\}) = 0$.

The converse holds for compact spaces and continuous maps.

Continuous selections

Theorem (MICHAEL, 1956) (follows from the abstract Lemma).

X paracompact, Y Fréchet, $\Phi: X \longrightarrow Y$ lower semicontinuous (i.e. $\mathfrak{V}$-$\mathfrak{V}$-measurable) correspondence with closed convex values, then Φ has a continuous selection.

Corollary. X paracompact, $Y \subseteq$ locally convex space E, compact convex metrizable, $\Phi: X \longrightarrow Y$ lower semicontinuous correspondence with closed convex values. Then Φ has a continuous selection.

Remark. Metrizability is essential (v. WEIZSÄCKER, 1975).

Applications

Paracompact spaces are characterized by the above selection property. If X, Y are as above, every continuous function $f: A \longrightarrow Y$ ($A \subseteq X$ closed) has a continuous extension to X.

Averaging operator in the sense of KELLEY

X compact metric, $\mathcal{A} \subseteq C(X)$ a subalgebra, R the equivalence relation on X induced by $\mathcal{A}$ ($x \sim y \Longleftrightarrow \forall a \in \mathcal{A}$: : $a(x) = a(y)$). Suppose the projection $\pi : X \longrightarrow X/_R$ is open and $X/_R$ is Hausdorff, then $\exists\ T: C(X) \longrightarrow \mathcal{A}$, $\|T\| = 1$, $T \geq 0$, $T^2 = T$ such that

$$\forall f \in C(X) \quad \forall a \in \mathcal{A} : T(f.a) = Tf.Ta \text{ (averaging equation).}$$

Using selections BLUMENTHAL/LINDENSTRAUSS/PHELPS (1965) show:

Theorem.

X compact metric, Y compact, $T: C(X) \longrightarrow C(Y)$ linear, $\|T\| \leq 1$. Then T is extreme iff $\exists\, \varphi : Y \longrightarrow X$ continuous, $\lambda \in C(Y)$, $\lambda^2 = 1$, such that $\forall f \in C(X) \quad Tf = \lambda . (f \circ \varphi)$.

Characterization of a class of compact convex sets.

$K \subseteq$ locally convex space E compact, convex, call K regular, iff there exists $\varphi : K \longrightarrow M_+^1(K)$ w^*-continuous, such that $\varphi(x)$ is a maximal representing measure for $x \in K$.

Theorem.

dim $K \leq 3$, extr K closed $\Longrightarrow$ K regular (K regular $\Longrightarrow$ extr K closed, always).

dim K = ∞ , extr K closed $\not\Longrightarrow$ K regular.

$\exists$ K regular, dim K = ∞ , K not a Bauer simplex (these are trivially regular).

K is regular iff a generalized Dirichlet problem is solvable, namely $\exists\, B \subseteq C(K)$ closed subspace with Choquet boundary extr K such that $\forall f \in C(\text{extr } K)$ $\exists\, Tf \in B$ such that $Tf|_{\text{extr } K} = f$, a affine $\Longrightarrow T(a)|_{\text{extr } K}) = a$ ($\Longrightarrow$ T linear, positive, isometric).

Metrizable CE-compact convex sets are regular (K is CE-compact iff the barycenter map $r: M_+^1(K) \longrightarrow K$ is open, LIMA, O'BRIEN)

E Banach space, $G \subseteq E$, $P_G(x) = \{y \in G \mid \|x - y\| = \|x - G\|\}$ (metric projection). If P_G is a correspondence (i.e. $P_G(x) \neq \emptyset \;\; \forall x$) the existence of continuous selections for this correspondence characterizes P_G for certain G (LAZAR/MORRIS/WULBERT, NÜRNBERGER) and in a certain sense the so called Lindenstrauss spaces (products of $L^1(\mu)$-spaces).