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RADON NIKODYM PROPERTY AND SET-VALUED INTEGRATION

by

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Let E be a Banach space, and E' be its conjugate. We denote by $\mathcal{C}(E)$, the set of closed bounded convex subsets of E. On $\mathcal{C}(E)$ we consider the following addition (denoted by $\dot{+}$)

$$C \dot{+} C' = \text{closure } (C + C')$$

We endow $\mathcal{C}(E)$ with its Hausdorff topology.
For $C \in \mathcal{C}(E)$, and $y \in E'$ we denote by $\delta^*(y/C)$ the scalar

$$\delta^*(y/C) = \sup \{\langle x,y\rangle \ / \ x \in C\} .$$

We denote by $\mathcal{K}(E)$, resp. $\mathcal{W}(E)$ the set of compact, resp. weakly compact convex subsets of E.

We consider a fixed complete positive finite measure space $(\Omega, \mathcal{T}, \mu)$.

Definition 1. Assume that E is separable. A map Γ from Ω to $\mathcal{C}(E)$ is said to be μ-measurable if one of the following equivalent conditions holds:

(i) There exists a sequence $(\sigma_n)_{n\geq 0}$ of measurable maps from Ω to E such that $\Gamma(\omega) = \text{closure}\{\sigma_n(\omega)/n \geq 0\}$ μ a.e.

(ii) The graph of $\Gamma = \{(\omega,x) \in \Omega \times E / x \in \Gamma(\omega)\}$ belongs to the product σ-algebra $\mathcal{T} \otimes$ (Borelians of E).

(The equivalence of (i) and (ii) is due to C. CASTAING.)

We call selection of Γ a measurable map $\sigma : \Omega \longrightarrow E$ such that: $\sigma(\omega) \in \Gamma(\omega)$ μ a.e.

We denote by $\mathcal{L}(\Gamma)$ the set of selections of Γ.

Definition 2. Let E be a separable Banach space, and Γ be a μ-measurable map from Ω to $\mathcal{C}(E)$. We say that Γ is μ-integrable if the following two properties are satisfied:

(i) For every $y \in E'$ the map $\omega \longrightarrow \delta^*(y/\Gamma(\omega))$ from Ω to $\mathbb{R}$ is μ-integrable

(ii) Every selection of Γ is Pettis-μ-integrable.

We denote $\int_A \Gamma \, d\mu$ the set = closure $\{\int_A \sigma \, d\mu / \sigma \in \mathcal{L}(\Gamma)\}$.

We have $\int_A \Gamma \, d\mu \in \mathcal{C}(E)$ for every $A \in \mathcal{T}$.

Theorem 1. Let $\Gamma : \Omega \longrightarrow \mathcal{C}(E)$ be μ-integrable, then the map M from $\mathcal{T}$ to $\mathcal{C}(E)$ defined by $M(A) = \int_A \Gamma \, d\mu$, $A \in \mathcal{T}$, satisfies the following properties.

(i) Whenever $A \cap B = \emptyset$, then $M(A \cup B) = M(A) \dot{+} M(B)$

(ii) Whenever $A = \bigcup_{\text{disjoint}} A_n$, then $M(A) = \dot{\sum\limits_{n \geq 0}} M(A_n)$,

i.e. this series is unconditionally convergent for the Hausdorff topology.

(iii) The variation $/M/$ of M is σ-finite.

(By definition $/M/(A) = \sup \{\sum_i \| x_i \| \, / (A_i)$ finite partition of A and $x_i \in M(A_i)\}$.)

(iv) For every $y \in E'$ we have:

$$\delta^*(y/\int_A \Gamma d\mu) = \int_A \delta^*(y/\Gamma(\omega))\,\mu(d\omega).$$

This last point is due to IOFFE-TIHOMIROV.

Definition 3. Let $(\Omega, \mathcal{T})$ be a measurable space. A map from $\mathcal{T}$ to $\mathcal{C}(E)$ is said to be a set-valued measure if it satisfies properties (i) and (ii) in Theorem 1.
We call selector of M a vector measure $m: \mathcal{T} \longrightarrow E$ such that:

$$m(A) \in M(A), \quad \forall A \in \mathcal{T}.$$

We denote by $\mathcal{S}(M)$ the set of selectors of M.

We say that M is rich if it satisfies the following property:

$M(A) = \text{Closure}\{m(A)/m \in \mathcal{S}(M)\}, \quad \forall A \in \mathcal{T}.$

Theorem 2. Let M be a set-valued measure from $\mathcal{T}$ to $\mathcal{C}(E)$.

1) If M is $\mathcal{W}(E)$-valued, then M is rich.

2) If E is separable, then M is rich.

3) If E has R.N.P., then M is rich.

(The point 1) is due to PALLU DE LA BARRIÈRE)

Problem 1: Is every set-valued measure rich ?

Definition 4. We say that a set-valued measure M from $\mathcal{T}$ to $\mathcal{C}(E)$ has a density with respect to μ, if there exists a μ-integrable map $\Gamma : \Omega \longrightarrow \mathcal{C}(E)$ such that $M(A) = \int_A \Gamma\, d\mu, \quad \forall A \in \mathcal{T}.$

Theorem 3. Let E be a separable Banach space having R.N.P. Then every set-valued measure M with σ-finite variation and absolutely continuous with respect to μ (i.e.

$\mu(A) = 0 \Longrightarrow M(A) = \{0\}$) has a density with respect to μ.

Question 1. Assume that in Theorem 3, M is $\mathcal{W}(E)$-valued. Is then the density of M also $\mathcal{W}(E)$-valued μ a.e. ?

Question 2. The same with $\mathcal{K}(E)$-valued.

The answer to question 2 is no (there exists a counter example in ℓ_2).

The answer to question 1 is yes if E' is separable, and no if $E \supset \ell_1$.

More generally we have the following theorem.

Theorem 4. Let E be a separable space such that E' is separable. Let $M: \mathcal{T} \longrightarrow \mathcal{W}(E)$ be a set-valued measure absolutely continuous with respect to μ, with σ-finite variation, and such that every selector m of M has a density with respect to μ (which is the case when E has R.N.P.). Then M has a $\mathcal{W}(E)$-valued density with respect to μ.

Let us call (P) the following property of a separable Banach space E:

Every set valued measure M with values in $\mathcal{W}(E)$ satisfying the assumptions of Theorem 4 has a $\mathcal{W}(E)$-valued density.

We know that:

E' separable $\Longrightarrow$ E satisfies (P) $\Longrightarrow E \not\supset \ell_1$

Problem 2: What is exactly property (P) ?