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by

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1. Introduction

Since Borsuk [1] first introduced the concept of shape in his study of the homotopy theory of compacta many authors (see, for example,[5,6,7,10,11,13,14,15,17]) have contributed to the development of shape theory. However the theory has remained almost exclusively confined to a topological context, never very far removed from the setting in which it was originally cast by Borsuk; and, further, and arising from this restriction in the scope of the theory, the concept has, in the work cited, related to some category of topological spaces $\mathfrak{T}$ and a full subcategory $\mathfrak{P}$ of $\mathfrak{T}$. However, Holsztynski [16] observed, soon after Borsuk's invention of the concept, that shape could be formulated as an abstract limit, and was thus of more general applicability.

It is the principal purpose of this paper to free shape theory from its restricted scope. Thus we replace the full embedding of a topological category $\mathfrak{P}$ in a topological category $\mathfrak{T}$ by an arbitrary functor $K : \mathfrak{P} \to \mathfrak{T}$ from the arbitrary category $\mathfrak{P}$ to the arbitrary category $\mathfrak{T}$. In so doing we are very much inspired by the point of view adopted by LeVan in his thesis [11]. We then find that many of the categorical aspects of shape theory (we do not speak of the topological aspects) remain valid in this very general setting. Others require some restriction on the functor K, but a restriction far milder than that K should be a full embedding.

We refer to the contribution of Sibe Mardešić to these proceedings for the foundations of shape theory. If $K : \mathfrak{P} \to \mathfrak{T}$ is the embedding of the homotopy category of compact polyhedra (or compact ANR's) in the homotopy category of compact Hausdorff spaces, then, basing himself on the Mardešić-Segal interpretation of Borsuk shape [15], via approximating ANR-systems, LeVan [11] showed the following. First, of course if $f : X \to Y$ is a map[1] in $\mathfrak{T}$, then f induces, for all objects P of $\mathfrak{P}$, a function $f^P : \mathfrak{T}(Y,P) \to \mathfrak{T}(X,P)$, simply by composition. Moreover, the functions f^P enjoy the naturality condition that, if $u : P \to Q$ is a map in $\mathfrak{P}$, then the diagram

(1.1)
$$\begin{tikzcd} \mathfrak{T}(Y,P) \arrow[r, "f^P"] \arrow[d, "u_*"] & \mathfrak{T}(X,P) \arrow[d, "u_*"] \\ \mathfrak{T}(Y,Q) \arrow[r, "f^Q"] & \mathfrak{T}(X,Q) \end{tikzcd}$$

commutes: here u_* is also induced by composition. Then LeVan's fundamental result in [11] is that a <u>shape morphism</u> from X to Y is nothing but a family of functions f^P, $P \in |\mathfrak{P}|$, such that (1.1) commutes for all $u : P \to Q$ in $\mathfrak{P}$. It is this point of view which we now adopt. Thus our generalization consists of replacing the special functor K by an arbitrary functor K between arbitrary categories $\mathfrak{P}$ and $\mathfrak{T}$ and <u>defining</u> the shape category by the obvious generalization of LeVan's characterization. Explicitly, given a functor K from a category $\mathfrak{P}$ to a category $\mathfrak{T}$, we define $\mathfrak{S}$, the <u>shape category of K</u>, to be the category whose objects are those of $\mathfrak{T}$, with (reexpressing (1.1)).

[1]Notice that a <u>map</u> is here a homotopy class of continuous functions.

$$(1.2) \qquad \mathfrak{S}(X,Y) = \mathrm{Nat}(\mathfrak{T}(Y,K-),\mathfrak{T}(X,K-))$$

Moreover, it is plain from the discussion above that every morphism of $\mathfrak{T}$ induces a morphism of $\mathfrak{S}$, so that there is a canonical functor $T : \mathfrak{T} \to \mathfrak{S}$ which is the identity on objects. Precisely, we regard the pair $(\mathfrak{S},T)$ as the shape of K.

Plainly, this generalization substantially broadens the scope of shape theory. However, it also has another purpose, namely, to identify those parts of the existing theory which are "trivial" - and to prove them by appropriately "trivial" arguments - and thus to enable one to focus, in any particular concretization, on the deep aspects of the theory. We will exemplify this latter aspect in the next section. Then in Section 3 we will apply shape theory in new contexts, thus exhibiting connections between different mathematical theories which are perhaps not immediately evident. We emphasize that the role of our categorical formulations is as stated above, and not to prove known or unknown difficult theorems. By means of our generalization we establish connections and know, as a result, what questions to ask in various mathematical contexts; to the "non-trivial" aspects of the answers we do not claim that our approach contributes.

Details of some of our specific results are to be found in [2,3].

2. Universal properties of shape theory

The approach taken in [14] shows that, in the original context of shape theory, we have the result

$$\mathfrak{T}(X,P) = \mathfrak{S}(X,P);$$

that is, the shape morphisms from a compact space X to a compact ANR P are just the original maps from X to P in $\mathfrak{T}$. It turns out that this property requires a mild restriction on the functor K, which leads to a concept which proves relevant in many contexts.

Definition 2.1 The functor $K : \mathfrak{P} \to \mathfrak{T}$ is rich [2] if, given objects P,Q of $\mathfrak{P}$ and a morphism $f : KP \to KQ$ in $\mathfrak{T}$, there exists a path

$$P = V_o \xrightarrow{v_1} V_1 \xleftarrow{v_2} V_2 \longrightarrow \cdots \xrightarrow{v_{2k-1}} V_{2k-1} \xleftarrow{v_{2k}} V_{2k} = Q \quad \text{in} \quad \mathfrak{P},$$

such that each Kv_{2i} is invertible and

$$f = (Kv_{2k})^{-1} o Kv_{2k-1} o \ldots o (Kv_2)^{-1} o Kv_1.$$

This definition is equivalent to the condition that, if we form the category of fractions $\mathfrak{P}[\Sigma^{-1}]$ with respect to the morphisms inverted by K, and if $\bar{K} : \mathfrak{P}[\Sigma^{-1}] \to \mathfrak{T}$ is induced by K, then $\bar{K}$ is full. An example of a rich functor which is not full is the direct limit functor from sequences of groups to groups.

Theorem 2.1 If $K : \mathfrak{P} \to \mathfrak{T}$ is rich then $T : \mathfrak{T}(X,KP) \to \mathfrak{S}(X,KP)$ is bijective for all X in $|\mathfrak{T}|$, P in $|\mathfrak{P}|$.

We would wish passage to the shape category to be idempotent. That is, if $K_1 = TK : \mathfrak{P} \to \mathfrak{S}$ we would wish $(\mathfrak{S},1)$ to be the shape of K_1. We find

Theorem 2.2 If $K : \mathfrak{P} \to \mathfrak{T}$ is rich then shape is idempotent

Indeed, as observed explicitly by A. Frei, the idempotence of shape follows from the conclusion of Theorem 2.1.

As a further example of a universal property of shape, consider the well-known result that, in the original restricted context of shape theory, Čech cohomology is shape-invariant. In our formulation, we say that a functor $G : \mathfrak{T} \to \mathfrak{C}$ is shape-invariant if it factors as $\bar{G}T$ with $\bar{G} : \mathfrak{S} \to \mathfrak{C}$. Plainly if G is shape-invariant then GX is equivalent to GY whenever X,Y have the same shape. Now Dold pointed out, in the appendix to [4], that Čech cohomology on the category of compact spaces is the right Kan extension [12] of ordinary (simplicial) cohomology on the category of compact polyhedra. Thus the shape-invariance of Čech cohomology is a special case of the following universal fact.

Theorem 2.3 Let $F : \mathfrak{P} \to \mathfrak{C}$ be a functor and let $\tilde{F} : \mathfrak{T} \to \mathfrak{C}$ be the right Kan extension of F along K. Then $\tilde{F}$ is shape-invariant.

In fact, there is a canonical factorization

$$\tilde{F} = \bar{F}T,\ \bar{F} : \mathfrak{S} \to \mathfrak{C},$$

and one easily proves

Theorem 2.4 If $K : \mathfrak{P} \to \mathfrak{T}$ is rich, then $\bar{F}$ is the right Kan extension of F along K_1 and the right Kan extension of $\tilde{F}$ along T.

Our final example of the universal aspect of shape theory is concerned with Grothendieck's notion of a pro-category. Let $\mathfrak{C}$ be a category and let $F : I \to \mathfrak{C}$, $G : J \to \mathfrak{C}$ be functors on (small) cofiltering categories I,J to $\mathfrak{C}$, then F,G are objects of the category Pro-$\mathfrak{C}$, and

$$(2.1) \qquad \text{Pro-}\mathfrak{C}(F,G) = \varprojlim_{j\in J} \varinjlim_{i\in I} \mathfrak{C}(Fi,Gj)$$

Now let $X{\downarrow}K$ be the comma category of $\mathfrak{P}$-objects under X, $X \in |\mathfrak{T}|$; and let $D_X : X{\downarrow}K \to \mathfrak{P}$ be the underlying functor given by

$$D_X(P,f) = P, \text{ where } f : X \to KP$$

$$D_X u = u, \text{ where } u : (P,f) \to (Q,g) \text{ in } X{\downarrow}K, \text{ that is,}$$

$$u : P \to Q \quad \text{and} \quad Ku \circ f = g$$

Then, as observed independently by K. Morita, in the original restricted context,

$$(2.2) \qquad \text{Pro-}\mathfrak{P}(D_X,D_Y) \cong \mathfrak{S}(X,Y)$$

However, one may show [3] that (2.2) continues to hold, virtually in complete generality. First we may take (2.1) as the definition of the pro-category even where the index categories (domains of F,G) are no longer cofiltering. This frees us of the necessity, in (2.2), of assuming - or, in any particular case such as the original context, proving - that the comma categories are cofiltering; and then (2.2) is universally true. Thus shape may, in general, be subsumed in the theory of (generalized) pro-categories.

3. Shape, localization and completion

Suppose now that $K : \mathfrak{P} \to \mathfrak{T}$ has a left adjoint $L : \mathfrak{T} \to \mathfrak{P}$. If $\eta : 1 \to KL$ is the unit of the adjunction, we may define a function $\Gamma : \mathfrak{S}(X,Y) \to \mathfrak{T}(X,KLY)$ by the rule

(3.1) $$\Gamma(\tau) = \tau^{LY}(\eta_Y)$$

Let Γ' consist of the composition of Γ and the adjunction-bijection $\mathfrak{T}(X,KLY) \cong \mathfrak{P}(LX,LY)$. One may then prove

<u>Theorem 3.1</u> $\Gamma' : \mathfrak{S}(X,Y) \to \mathfrak{P}(LX,LY)$ <u>is bijective and respects identities and composition</u>. <u>Thus</u> $\mathfrak{S}$ <u>is isomorphic to the Kleisli category of</u> $\mathfrak{T}$ <u>with respect to the triple generated by the adjunction</u> $L \dashv K$.

This theorem implies that, when K admits a left adjoint L, then we may regard a shape morphism from X to Y as an ordinary ($\mathfrak{T}$-)morphism from X to KLY. Moreover given a shape morphism from X to Y, i.e., $f : X \to KLY$, and a shape morphism from Y to Z, i.e., $g : Y \to KLZ$, we compose them, to produce a morphism $h : X \to KLZ$ by the rule

$$h = Kg' \circ f,$$

where g' corresponds to g under the adjunction-bijection $\mathfrak{T}(Y,KLZ) \cong \mathfrak{P}(LY,LZ)$.

As an example of this theorem, consider the following. Let P be a family of prime numbers, let $\mathfrak{T}$ be the category $\mathfrak{N}$ of nilpotent groups and let $\mathfrak{P}$ be the full subcategory $\mathfrak{N}_P$ consisting of P-local nilpotent groups. Then it is known (see, e.g. [8,9]) that the full embedding $K : \mathfrak{N}_P \to \mathfrak{N}$ has a left adjoint L. It is customary to write G_P for LG (or KLG), $G \in |\mathfrak{N}|$, so that

$$\mathfrak{S}(G,H) = \mathrm{Hom}(G,H_P).$$

The localizing map $e : H \to H_P$ is the unit of the adjunction so that a homomorphism $\psi : H \to K_P$ in $\mathfrak{N}$ determines a unique

$\psi_P : H_P \to K_P$ such that $\psi_P e = \psi$. Then we compose $\varphi \in \mathfrak{S}(G,H)$, $\psi \in \mathfrak{S}(H,K)$, that is, $\varphi : G \to H_P$, $\psi : H \to K_P$, to produce $\psi_P\varphi : G \to K_P$.

The notion of richness again enters the story at this point. For one may prove

<u>Proposition 3.2</u> <u>Let</u> $K : \mathfrak{P} \to \mathfrak{T}$ <u>admit the left adjoint</u> $L : \mathfrak{T} \to \mathfrak{P}$. <u>Then the following statements are equivalent</u>:

(i) <u>The triple generated by the adjunction is idempotent</u>;

(ii) <u>K is rich</u>;

(iii) <u>L is rich</u>.

It follows that, if K is rich, then, for all Y in $|\mathfrak{T}|$, KLY is the Adams completion of Y with respect to the morphisms of $\mathfrak{T}$ inverted by L, thus

$$\mathfrak{T}[\Sigma_L^{-1}](-,Y) \cong \mathfrak{T}(-,KLY)$$

Combining this with Theorem 3.1 we have

<u>Theorem 3.3</u> <u>If</u> $K : \mathfrak{P} \to \mathfrak{T}$ <u>is a rich functor admitting a left adjoint</u> L, <u>then</u> $\mathfrak{S} \cong \mathfrak{T}[\Sigma_L^{-1}]$, <u>where</u> Σ_L <u>is the family of morphisms inverted by</u> L.

Now it is easy to see that the family of morphisms of $\mathfrak{T}$ inverted by L coincides with the family of morphisms of $\mathfrak{T}$ inverted by $T : \mathfrak{T} \to \mathfrak{S}$. It is thus reasonable to propose the following question.

<u>Question</u> Suppose $K : \mathfrak{P} \to \mathfrak{T}$ is rich. When is $\mathfrak{S}$ the category of fractions $\mathfrak{T}[\Sigma_T^{-1}]$, where Σ_T is the family of morphisms inverted by T?

It is interesting to note that, when the answer is affirmative, and when the Adams completion Y_T of Y in $|\mathfrak{T}|$ exists, then $\mathfrak{S}(X,Y) = \mathfrak{T}(X,Y_T)$. Thus we are motivated to look for examples (when K does <u>not</u> admit a left adjoint) when the shape morphisms from X to Y are ordinary morphisms from X to some appropriate "modification" of Y.

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