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On an Extension of Pontryagin's Duality Theory

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1. Introduction

Let G be a commutative, topological group. A character of G is a continuous homomorphism $h : G \longrightarrow S^1$, where the group S^1 is the compact group of all complex numbers of modulus one. Now let G be locally compact. The collection ΓG of all characters of G, endowed with the topology of compact convergence, forms a commutative, locally compact, topological group $\Gamma_c G$ under the pointwise defined operations. In addition, the natural homomorphism

$$j_G : G \longrightarrow \Gamma_c \Gamma_c G \quad ,$$

defined by $j_G(g)(\gamma) = \gamma(g)$ for each $g \in G$ and each $\gamma \in \Gamma_c G$, is, as the fundamental theorem of Pontryagin states, a bicontinuous isomorphism. Pontryagin's duality theory is the study of the rich relations between G and $\Gamma_c G$.

The aim of this note is to suggest an extension of Pontryagin's duality theory by extending the fundamental theorem to a wider class of groups. We proceed as follows: To the (commutative) groups under consideration will be associated a concept of convergence compatible with the algebraic structure. Groups of this sort are called convergence groups. This concept of convergence, given by a convergence structure, will allow the notion of continuity. For any convergence group G, the group ΓG of all characters of G equipped with the continuous convergence structure Λ_c will be denoted by $\Gamma_c G$. In case G is a locally compact topological group, Λ_c is identical to the topology of compact convergence. We will call a convergence group

P_c-reflexive, if $j_G : G \longrightarrow \Gamma_c\Gamma_c G$ is a bicontinuous isomorphism. The class of P_c-reflexive convergence groups contains in addition to all commutative, locally compact, topological groups all the complete, locally convex $\mathbb{R}$-vector spaces. We will verify the P_c-reflexivity of the following type of topological groups:

For any $k = o,\dots,\infty$ the collection $C^k(M,S^1)$ of all S^1-valued C^k-functions of a connected compact C^∞-manifold M, equipped with the C^k-topology is a topological group. It is in general not locally compact. We demonstrate the P_c-reflexivity of $C^k(M,S^1)$ as follows:

The idea is to use $C^k(M)$, the complete, locally convex vector space of real-valued C^k-functions of M equipped with the C^k-topology and to introduce $C^k(M)/\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ denotes the subset of all the functions assuming their values in $\mathbb{Z}$. We will show, that $C^k(M)/\underline{\mathbb{Z}}$ can be identified with the connected component $\varkappa_M C^k(M)$ of $\underline{1}$ in $C^k(M,S^1)$. The quotient $C^k(M,S^1)/\varkappa_M C^k(M)$ is then a discrete group called $\Pi^1(M)$. The exact sequence

$$\underline{1} \longrightarrow \varkappa_M C^k(M) \longrightarrow C^k(M,S^1) \longrightarrow \Pi^1(M) \longrightarrow 1$$

has an exact "bidual":

$$\underline{1} \longrightarrow \Gamma_c\Gamma_c\varkappa_M C^k(M) \longrightarrow \Gamma_c\Gamma_c C^k(M,S^1) \longrightarrow \Gamma_c\Gamma_c\Pi^1(M) \longrightarrow \underline{1}.$$

Since $\varkappa_M C^k(M)$ and $\Pi^1(M)$ will turn out to be P_c-reflexive, we will conclude, via the five lemma, that $C^k(M,S^1)$ is also P_c-reflexive. Along the way, we will study some special character groups appearing in our procedure.

For this type of extension of Pontryagin's duality theory a suitable extension theorem of characters is still missing. This hinders considerably the study of the relations between G and $\Gamma_c G$ for P_c-reflexive convergence groups G.

2. Review of some Definitions and Results

2.1. The character group of a convergence group, P_c-reflexivity

Let X be a non empty set. To any point in X will be associated a collection $\Lambda(p)$ of filters on X. The set $\Lambda(p)$ is an element of $P(F(X))$, the power set of the set of all filters $F(X)$ of X.

The map $\Lambda : X \longrightarrow P(F(X))$ is called a convergence structure on X if the following conditions are satisfied for each $p \in X$:

(i) $\dot{p}$, the filter generated by $\{p\}$ belongs to $\Lambda(p)$.

(ii) Any filter finer than a member of $\Lambda(p)$ belongs to $\Lambda(p)$.

(iii) The infimum $\Phi\wedge\psi$ of any two filters of $\Lambda(p)$ belongs to $\Lambda(p)$.

Let us remark here, that any topology on X is a convergence structure, but not vice versa.

The set X, together with a convergence structure Λ, is called a convergence space. The filters in $\Lambda(p)$ are said to converge to p in X. A map f from a convergence space X into a convergence space Y is continuous if, for any filter Φ convergent to p in X, the image converges to f(p) in Y. The cartesian product $X \times X$ of any two convergence spaces X and Y carries the product structure defined in the obvious way [Bi].

On C(X,Y), the collection of the continuous maps from the convergence space X into the convergence space Y, there is a coarsest among all the convergence structures for which the evaluation map

$$\omega : C(X,Y) \times X \longrightarrow Y ,$$

(defined by $\omega(f,p) = f(p)$ for any $(f,p) \in C(X,Y) \times X$) is continuous. This is called the continuous convergence structure Λ_c . A filter Θ on C(X,Y) converges to a function f with respect to Λ_c iff for any $p \in X$ the filter $\omega(\Theta \times \Phi)$ converges to f(p) in Y for any filter Φ convergent to p. The set $C(X,Y)$ and any subset $A(X,Y)$ of $C(X,Y)$ endowed with Λ_c are denoted by $C_c(X,Y)$ and $A_c(X,Y)$ respectively. The continuous convergence structure is characterized by the following universal property ([Bi], [Bi,Ke]) : A map f from a convergence space S into a subspace $A_c(X,Y)$ of $C_c(X,Y)$ is continuous iff $\omega \circ (f \times id) : S \times X \longrightarrow Y$ is continuous.

We now pass on to convergence groups. Our groups are always assumed to be abelian.

A group, together with a convergence structure, is called a <u>convergence group</u> if the group operations are continuous.

The character group $\Gamma_c G$ of a convergence group G is the group ΓG of all continuous homomorphisms of G into the circle group S^1 together with the continuous convergence structure. The operations on ΓG are defined pointwise. Obviously, $\Gamma_c G$ is a convergence group. The canonical map

$$j_G : G \longrightarrow \Gamma_c \Gamma_c G ,$$

defined by $j_G(g)(\gamma) = \gamma(g)$ for any $g \in G$ and any $\gamma \in \Gamma_c G$ is evidently continuous.

We call G P_c-<u>reflexive</u> if j_G is a bicontinuous isomorphism

<u>Remark:</u> If G is a locally compact topological group, then the continuous convergence structure on ΓG is identical to the topology of compact convergence. Hence the P_c-reflexivity of such a group G is identical to the classical reflexivity in the sense of Pontryagin [Po].

2.2 The character group of a convergence vector space

An $\mathbb{R}$-vector space E (referred to as a vector space) equipped with a convergence structure for which the operations are continuous is called a <u>convergence vector space</u> [Bi]. The <u>c-dual</u>, $\mathcal{L}_c E$, of E

is the vector space of all continuous real-valued linear functionals endowed with the continuous convergence structure.

The exponential map from $\mathbb{R}$ to S^1 sending each real r to $e^{2\pi ir}$ is denoted by $\varkappa$. This map induces a continuous homomorphism $\varkappa_* : \mathcal{L}_c E \longrightarrow \Gamma_c E$ assigning to each $\ell \in \mathcal{L}_c E$ the character $\varkappa \circ \ell$. It is shown in [Bu], that $\varkappa_*$ is a bicontinuous isomorphism. For a slightly restricted version of this result, which is general enough for our purposes, we refer the reader to the Appendix in [Bi]. The proof of the result in [Bu] is an elaborated version of the proof I gave in [Bi]. For an earlier result in this direction see [F-S].

Let us point out here, that there is no vector space topology T on $\mathcal{L}E$, where E is locally convex, for which the evaluation map $\omega : \mathcal{L}E \times E \longrightarrow \mathbb{R}$ is continuous, unless E is normable [Ke].

We call a convergence vector space E c-reflexive if $i_E : E \longrightarrow \mathcal{L}_c \mathcal{L}_c E$ is a bicontinuous isomorphism.

One easily verifies [Bi]:

Lemma 1: A convergence vector space E is P_c-reflexive iff E is c-reflexive. A topological vector space E is P_c-reflexive iff it is locally convex and complete.

2.3. P_c-reflexivity of some convergence groups of continuous mappings

Assume that X is an arcwise connected topological space. The map $\varkappa_X : C_c(X) \longrightarrow C_c(X,S^1)$, sending each $f \in C_c(X)$ into $\varkappa \circ f$, is a quotient map onto its range, regarded as a subspace of $C_c(X,S^1)$ [Bi,2]. The quotient $C_c(X,S^1)/\varkappa_X C_c(X)$, carrying the quotient structure is denoted by $\Pi_c^1(X)$ and is called [Hu] the Bruschlinski group of X. If X is locally compact, $C_c(X,S^1)$ is a topological group. As demonstrated in [Bi,2], we have:

Theorem 2 The group $\varkappa_X C_c(X)$ is P_c-reflexive. If, in addition, X is a normal space allowing a (simply connected) universal covering, the group $C_c(X,S^1)$ is P_c-reflexive if $\Pi_c^1(X)$ is complete. This is the case e.g. if either the first singular homology group (with the integers as coefficients) is finitely generated, or $\Pi^1(X)$ is isomorphic to the first singular cohomology group (with the integers as coefficients).

In the next few sections we will derive some functional analytic results which will, in turn, be fundamental in showing the P_c-reflexivity of $C^k(M,S^1)$, as announced in the introduction.

3. Functional analytic preliminaries

3.1. $C^k(M)$ for a connected compact C^∞-manifold M

Let M be a compact C^∞-manifold. For a non-negative integer k, we will denote by $C^k(M)$ the Banach space of all real-valued C^k-functions of M, endowed with the usual norm. This yields the topology of uniform convergence in all k derivatives. We refer to [Pa] and [Go,Gui] for the above remarks and for the next few details. Clearly the inclusion map $j_k^{k+1} : C^{k+1}(M) \longrightarrow C^k(M)$ is continuous for any k. Moreover, its image is dense and the image of the unit ball E_{k+1} of $C^{k+1}(M)$ is relatively compact in $C^k(M)$.

The projective limit of all $C^k(M)$ is denoted by $C^\infty(M)$. This is a complete, metrizable, locally convex space, a so-called Fréchet space [Schae]. Since E_{k+1} is relatively compact in $C^k(M)$ for any k, the space $C^\infty(M)$ is called a Schwartz space.

3.2. The c-dual of $C^k(M)$

First, let F be any convergence vector space. Any <u>compact</u> set in $\mathcal{L}_c F$ is <u>topological</u> [Bi]. A convergence space is said to be <u>compact</u> if every ultrafilter converges to exactly one point.

Next, we describe the c-dual of F where F is a <u>topological</u> vector space. For any neighborhood U of zero in F, the polar $\{\ell \in LF \mid \ell(U) \subset [-1,1]\}$, denoted by U^o, is compact if regarded as a subspace of L_cF. Hence it is a compact topological space. The topology on it is the topology of pointwise convergence. Moreover, L_cF is the inductive limit (in the category of convergence spaces) of all these compact topological spaces U^o, where U runs through the neighborhood filter of zero in F. For these and the next few details we refer the reader to [Bi] or to [Bi,Bu,Ku].

For a topological vector space F, the natural map $i_F : F \longrightarrow L_cL_cF$ is a bicontinuous isomorphism iff F is a complete,locally convex vector space (cf. Lemma 1).

Let us turn our attention to $L_cC^k(M)$ for a finite k. The convergence vector space $L_cC^k(M)$ is the inductive limit of all multiples of the polar E_k^o of the unit ball $E_k \subset C^k(M)$. Here as a subspace of $L_cC^k(M)$, E_k^o carries the topology of pointwise convergence and is therefore compact.

When $LC^k(M)$ carries the usual norm topology, we write $L_nC^k(M)$.

For any k, the adjoint of j_k^∞, the map

$$j_k^{\infty *} : L_cC^k(M) \longrightarrow L_cC^\infty(M) \ ,$$

defined by composing each $\ell \in L_cC^k(M)$ with j_k^∞, is a continuous injection. Since $C^\infty(M)$ is a Schwartz space, we even have [Ja] :

<u>Lemma 3</u> $L_cC^\infty(M)$ is the inductive limit (in the category of convergence spaces) of $L_cC^k(M)$ as well as of $L_nC^k(M)$, taken over all finite k.

For any $p \in M$, the linear functional $i_M(p) : C^k(M) \longrightarrow \mathbb{R}$ evaluating each $f \in C^k(M)$ at p is continuous for any k. If $k < \infty$ $i_M^k : M \longrightarrow L_cC^k(M)$ sending each $p \in M$ into $i_M^k(p)$ is a continuous injection whose image is contained in the polar E_k^o of the unit ball $E_k \subset C^k(M)$. Hence we have:

Lemma 4: The canonical map $i_M^k : M \longrightarrow L_c C^k(M)$ is (for any k) a homeomorphism onto a subspace of $L_c C^k(M)$. If $k < \infty$, then $i_M^k(M) \subset E_k^o$.

3.3. $V^k(M)$

For each $k=o,1,\ldots,\infty$ let $V^k(M)$ be the span of $i_M^k(M)$, regarded as a subspace of $L_c C^k(M)$.

Recall that in a convergence space X a point p is adherent to a subset A if there is a filter Φ convergent to p in X, such that $F \cap A \neq \emptyset$ for any $F \in \Phi$. We call $A \subset X$ dense if the collection $\bar{A}$ (the adherence of A) of all points adherent to A is all of X.

The following is an analogue to the situation in the case of $C_c(X)$ (cf. appendix of [Bi]).

Theorem 5: The space $V^\infty(M)$ is dense in $L_c C^\infty(M)$. Moreover the restriction map $r^\infty : L_c L_c C^\infty(M) \longrightarrow L_c V^\infty(M)$ is a bicontinuous isomorphism. Thus $\rho^\infty : L_c V^\infty(M) \longrightarrow C^\infty(M)$, defined by $\rho^\infty(\ell) = \ell \circ i_M^\infty$ for each $\ell \in L_c V^\infty(M)$, is a bicontinuous isomorphism.

Proof: Since $C^\infty(M)$ is c-reflexive, r^∞ is a monomorphism. To show its surjectivity, consider for each finite k the following diagram:

$$V^\infty(M) \xleftarrow{\ a\ } V_n^{k+1}(M) \subset L_n C^{k+1}(M) \xleftarrow{(j_k^{k+1})^*} L_c C^k(M) .$$

The index n indicates, that the respective spaces carry the usual norm topology. The linear map a sending each $i_M^{k+1}(p)$ into $i_M^\infty(p)$, is evidently continuous. Finally $(j_k^{k+1})^*$ restricts each $\ell \in L_c C^k(M)$ to $C^{k+1}(M)$. The next goal is to show that $(j_k^{k+1})^*$ is continuous. We recall that the unit ball E_{k+1} of $C^{k+1}(M)$ is relatively compact in $C^k(M)$. Hence the polar E_k^o of E_k formed in $LC^k(M)$ is mapped by $(j_k^{k+1})^*$ into a compact subspace $(j_k^{k+1})^*(E_k^o)_n$ of the Banach space $L_nC^{k+1}(M)$ (cf. [Schae] p.111). From this, we conclude the continuity of $(j_k^{k+1})^*$. Next let $\ell \in L_cV^\infty(M)$. The functional $\ell \circ a$ has a continuous extension $\bar{\ell}$ to $L_nC^{k+1}(M)$, for which $\bar{\ell}\circ(j_k^{k+1})^* = \ell'$ is continuous on $L_cC^k(M)$. Moreover $\ell' \circ i_M^k = \ell \circ i_M^\infty$. Since $C^k(M)$ is c-reflexive, ℓ' can be represented as $i_{C^k(M)}(f_k)$ for some function $f_k \in C^k(M)$. Hence $\ell' \circ i_M^k = f_k$ for each finite k. Since $\ell \circ i_M^\infty = f_k$, the function $\ell \circ i_M^\infty$ is of class C^∞ and $r^\infty \circ i_{C^\infty(M)}(f_k) = \ell$. Thus the injection is bijective. We proceed now to show that $V^\infty(M)$ is dense in $L_cC^\infty(M)$. To do this, we introduce $\overline{V^\infty(M)}'$ and establish three properties (a,b,c) which exhibit this space as an L_c-space [Bi,Bu,Ku]. Thus $\overline{V^\infty(M)}'$ is c-reflexive. Let us point out that $L_cC^\infty(M)$ is the inductive limit (in the category of convergence spaces) of countably many absolutely convex, compact topological spaces $K_1 \subset K_2 \subset \ldots$. Hence we have $\bigcup_i K_i = LC^\infty(M)$. For each index i we form the adherence $\overline{K_i \cap V^\infty(M)}^i$ of $K_i \cap V^\infty(M)$ in K_i . This adherence is a convex, compact topological subspace of K_i . Moreover

$$\overline{V^\infty(M)} = \bigcup_i \overline{K_i \cap V^\infty(M)}^i ,$$

as one easily verifies. Hence $\overline{V^\infty(M)}'$, regarded as the inductive limit of the compact, convex subspaces $\overline{K_i \cap V^\infty(M)}^i$, taken over all i, is a) locally convex, and locally compact and b) admits point-separating continuous linear functionals. By locally convex we mean that, for any filter convergent to q , there is a coarser one having a basis of convex sets which also converges to q. Locally compact means that any convergent filter contains a compact set.

The last one, c), of the above mentioned characteristic properties is the following: Any compact subspace of $\overline{V^\infty(M)}'$ is a compact topological space. But this is evidently true because any compact subset of $\overline{V^\infty(M)}'$ is contained in one of the compact topological spaces

$\overline{K_i \cap V^\infty(M)}^i$. Thus $\overline{V^\infty(M)}'$ is an L_c-space. Since $\overline{V^\infty(M)}'$ splits into countably many compact subsets, $L_c\overline{V^\infty(M)}'$ is a Fréchet space. *Hence it is c-reflexive [Bi,Bu,Ku]. In addition,* $V^\infty(M)$ *is a dense subspace of* $\overline{V^\infty(M)}'$. *One easily concludes that*

$$r^\infty : L_cL_cC^\infty(M) \longrightarrow L_c\overline{V^\infty(M)}'$$

is a continuous bijection between Fréchetspaces. Using the closed graph theorem, we deduce that r^∞ *is a homeomorphism.* The c-reflexivity of $\overline{V^\infty(M)}'$ *and* $C^\infty(M)$ *now immediately yields* $\overline{V^\infty(M)} = L_cC^\infty(M)$. The commutativity of

$$\begin{array}{ccccc} L_cL_cC^\infty(M) & \xrightarrow{\;r^\infty\;} & L_cV^\infty(M) & \xrightarrow{\;\rho^\infty\;} & C^\infty(M) \\ \uparrow{\scriptstyle i_{C^\infty(M)}} & & & \nearrow{\scriptstyle \mathrm{id}} & \\ C^\infty(M) & & & & \end{array}$$

allows us to conclude that r^∞ and ρ^∞ are bicontinuous isomorphisms, as asserted in theorem 5.

4. $\varkappa_M C^k(M)$, in particular $\varkappa_M C^\infty(M)$

4.1 The group $\varkappa_M C^k(M)$ and its P_c-reflexivity

For any $k = 0,\dots,\infty$, we consider the collection $\varkappa_M C^k(M)$ of all functions $\varkappa \circ f$, where $f \in C^k(M)$. (Recall, that $\varkappa : \mathbb{R} \longrightarrow S^1$ sends each r into $e^{2\pi i r}$). This collection is a group under the pointwise defined operations. Since M is connected, the kernel of

$$\varkappa_M : C^k(M) \longrightarrow \varkappa_M C^k(M)$$

is $\underline{\mathbb{Z}}$, the subgroup of all constant functions assuming their values in $\mathbb{Z}$. For any $z \in \mathbb{Z}$ denote by $\underline{z}$ the function whose only value is z . By virtue of the addition in $C^k(M)$, $\mathbb{Z}$ operates on $C^k(M)$ properly discontinuously [Spa]. Hence the quotient $C^k(M)/\underline{\mathbb{Z}}$, taken in the category of topological spaces, has $C^k(M)$ as its (simply connected) universal covering. From this, we conclude that $C^k(M)/\mathbb{Z}$ is also the quotient in the category of convergence spaces. Moreover $\mathbb{Z}$

is isomorphic to the fundamental group of $C^k(M)/\underline{\mathbb{Z}}$. Let us identify $C^k(M)/\underline{\mathbb{Z}}$ with $\varkappa_M C^k(M)$ and the projection map onto $C^k(M)/\underline{\mathbb{Z}}$ with $\varkappa_M$.

The topological group $\varkappa_M C^k(M)$ can be represented as a direct product (I thank H.P.Butzmann for reminding me of this fact): To a given point $p \in M$ consider the subspace $m_p^k \subset C^k(M)$ consisting of all C^k-functions vanishing on p. For any two functions $f_1, f_2 \in m_p^k$ we have $f_1 - f_2 \notin \underline{\mathbb{Z}}$ unless they are identical. Hence $\varkappa_M | m_p^k$ is an injection and we conclude from the topological direct sum decomposition $C^k(M) = m_p^k \oplus \mathbb{R} \cdot \underline{1}$ that

$$\varkappa_M C^k(M) = \varkappa_M(m_p^k) \cdot S^1 \cdot \underline{1} .$$

This direct decomposition is evidently topological. Since $m_p \subset C^k(M)$ is a complete, locally convex topological vector space, it is P_c-reflexive (Lemma 1). Since S^1 is also P_c-reflexive, $\varkappa_M C^k(M)$ is P_c-reflexive. Thus we have:

<u>Theorem 6</u> For any $k = o, \dots, \infty$, the topological group $\varkappa_M C^k(M)$ has $C^k(M)$ as its universal covering with a fundamental group isomorphic to $\mathbb{Z}$, splits topologically into

$$\varkappa_M C^k(M) = \varkappa_M(m_p^k) \cdot S^1 \cdot \underline{1} .$$

and is thus P_c-reflexive.

4.2. The character group of $\varkappa_M C^\infty(M)$; the group $P_c^\infty(M)$

A linear combination $\Sigma r_i \cdot i_M^\infty(p_i) \in V^\infty(M)$ composed with $\varkappa$ factors through $\varkappa_M$ iff $\Sigma r_i \in \mathbb{Z}$. Denote by $P_c^\infty(M)^1 \subset \Gamma_c C^\infty(M)$ the collection of all combinations of the form $\varkappa \circ \Sigma r_i \cdot i_M^\infty(p_i)$ for which $\Sigma r_i \in \mathbb{Z}$, equipped with the continuous convergence structure. Since $\varkappa_M$ is a quotient map, the continuous homomorphism

$$\bar{\varkappa} : P_c^\infty(M)^1 \longrightarrow \Gamma_c \varkappa_M C^\infty(M) \quad ,$$

assigning to each character in $P_c^\infty(M)^1$ its factorization through $\varkappa_M$, is a bicontinuous isomorphism onto a convergence subgroup of $\Gamma_c \varkappa_M C^\infty(M)$.

Denoting this convergence subgroup by $P_c^\infty(M)$, then we have a bicontinuous isomorphism

$$\bar{\varkappa} : P_c^\infty(M)^1 \longrightarrow P_c^\infty(M) .$$

<u>Lemma 7</u> $P_c^\infty(M)$ is dense in $\Gamma_c \varkappa_M C^\infty(M)$.

The proof is analogous to that of Lemma 8 (p.67) given in [Bi,2] .

We may reformulate Lemma 7 by saying that the character group $\Gamma_c \varkappa_M C^k(M)$ of $\varkappa_M C^k(M)$ is generated by $P_c^\infty(M)$.

Next consider the injective mapping

$$j_M^\infty : M \longrightarrow P_c^\infty(M) \subset \Gamma_c \varkappa_M C^\infty(M)$$

defined by $j_M^\infty(p)(t) = t(p)$ for all $p \in M$ and all $t \in \varkappa_M C^\infty(M)$. Since $\varkappa_M$ is a quotient map, we conclude by Lemma 4, that j_M^∞ maps M homeomorphically onto a subspace of $P_c^\infty(M)$. Any character $\gamma \in \Gamma P_c^\infty(M)$ induces an S^1-valued function $\gamma \circ j_M^\infty$.

<u>Lemma 8</u> For each $\gamma \in \Gamma P_c^\infty(M)$ the function $\gamma \circ j_M^\infty$ belongs to $\varkappa_M C^\infty(M)$. The map

$$\varphi : \Gamma_c P^\infty(M) \longrightarrow \varkappa_M C^\infty(M)$$

sending each γ into $\gamma \circ j_M^\infty$ is a continuous monomorphism.

<u>Proof:</u> For $\gamma \in \Gamma P_c^\infty(M)$ consider $\gamma \circ \bar{\varkappa} \in \Gamma P_c^\infty(M)^1$; and denote $\varkappa_*^{-1}(P_c^\infty(M)^1)$ by $V_1^\infty(M) \subset L_c C^\infty(M)$. Pulling the character $\gamma \circ \bar{\varkappa}$ back onto $V_1^\infty(M)$, we obtain the character $\gamma \circ \bar{\varkappa} \circ (\varkappa_* | V_1^\infty(M)) : V_1^\infty(M) \longrightarrow S^1$. Our aim is to extend this character onto the whole space $V^\infty(M)$ and then (using theorem 5) to show that $\gamma \circ j_M^\infty$ is of class C^∞. For this purpose we decompose $V^\infty(M)$ as follows: One factor is M_o , the kernel of the linear functional $i_{C^\infty(M)}(\underline{1}) : V^\infty(M) \longrightarrow \mathbb{R}$, sending each linear combination $\Sigma r_i \cdot i_M^\infty(p_i)$ into Σr_i. Hence M_o consists of all linear combinations $\Sigma_i r_i \cdot i_M^\infty(p_i)$ with $\Sigma r_i = o$. For a fixed point $p \in M$,

we form $\mathbb{R}\cdot i_M^\infty(p)$, which is homeomorphic to $\mathbb{R}$. One easily shows now that

$$\text{(i)} \qquad V^\infty(M) = M_o \oplus \mathbb{R}\cdot i_M^\infty(p)$$

holds as an identity between convergence vector spaces. Hence $V_1^\infty(M) \subset V^\infty(M)$ decomposes as

$$\text{(ii)} \qquad V_1^\infty(M) = M_o \oplus \mathbb{Z}\cdot i_M^\infty(p) .$$

(An analoguous decomposition holds for any k.) We therefore split $\gamma \circ \bar{\varkappa} \circ (\varkappa_* | V_1^\infty(M))$ into the product $\gamma_1 \cdot \gamma_2$ of its restrictions $\gamma_1 = \gamma \circ \bar{\varkappa} \circ (\varkappa_* | M_o)$ and $\gamma_2 = \gamma \circ \bar{\varkappa} \circ (\varkappa_* | \mathbb{Z}\cdot i_M^\infty(p))$. Using the classical extension theorem of characters, we extend $\gamma_2 : \mathbb{Z}\cdot i_M^\infty(p) \longrightarrow S^1$ to $\bar{\gamma}_2 : \mathbb{R}\cdot i_M^\infty(p) \longrightarrow S^1$. Then $\gamma_1 \cdot \bar{\gamma}_2$ is a continuous character on $V^\infty(M)$ which corresponds via $\varkappa_*$ to a continuous linear functional $\ell \in L_c V^\infty(M)$. By theorem 5, the functional ℓ is of the form $(\rho^\infty)^{-1}(f)$ where $f \in C^\infty(M)$. From this we conclude $\varphi(\gamma) = \varkappa_M(f)$. Since ℓ is uniquely determined by its values on $i_M^\infty(M) \subset V^\infty(M)$, the continuous map φ is a monomorphism. This completes the proof. (The methods used above yield simplifications in the proof of Satz 7, p.62 in [Bi,2].)

Finally, let us collect some of our results on $\varkappa_M C^\infty(M)$ and its character group in the following theorem.

<u>Theorem 9</u> The topological group $\varkappa_M C^k(M)$ splits topologically into $\varkappa_M(m_p^k)\cdot S^1 \cdot \underline{1}$ where $m_p^k \subset C^k(M)$ consists of all C^k-functions vanishing on a fixed point $p \in M$. The character group of $\varkappa_M C^\infty(M)$ is generated by $P_c^\infty(M)$. Moreover $\varphi : \Gamma_c P_c^\infty(M) \longrightarrow \varkappa_M C^\infty(M)$, sending each character γ into $\gamma \circ j_M^\infty$, is a bicontinuous isomorphism. In addition, $P_c^\infty(M)$ splits topologically into $\bar{\varkappa}(N_o)\cdot \mathbb{Z} \cdot j_M(p)$, where N_o carries the continuous convergence structure and consists of all combinations $\varkappa \circ \Sigma r_i \cdot i_M^\infty(p_i) \in \Gamma C^\infty(M)$ with $\Sigma r_i = o$. The character group of $\bar{\varkappa}(N_o)$ is bicontinuously isomorphic to $\varkappa_M(m_p^\infty)$.

<u>Proof:</u> The first two assertions are valid by theorem 1 and Lemma 7. To verify the others, consider the commutative diagram of continuous maps:

$$\begin{array}{ccccc} \Gamma_c\Gamma_c\varkappa_M C^\infty(M) & \longrightarrow & \Gamma P^\infty(M) & \xrightarrow{\varphi} & \varkappa_M C^\infty(M) \\ \uparrow j^\infty_{\varkappa_M C^\infty(M)} & & & \nearrow \text{id} & \\ \varkappa_M C^\infty(M) & & & & \end{array}$$

where the first horizontal arrow indicates the restriction map. Using this diagram in combination with Lemmas 7 and 8, we easily obtain the bijectivity of φ, the continuity of φ^{-1} and thus the P_c-reflexivity of $\varkappa_M C^\infty(M)$ again. That $P_c^\infty(M)$ splits into $\bar{\varkappa}(N_o)\cdot\mathbb{Z}\cdot j^k_M(p)$ is evident by using (ii) in the proof above and $\bar{\varkappa}$ introduced at the beginning of section 4.2. The rest of the theorem is straightforward.

5. $C^k(M,S^1)$ and its P_c-reflexivity

5.1. The Bruschlinski group

The collection of all S^1-valued C^k-functions endowed with the C^k- topology [Go,Gui] forms a topological group under the pointwise defined operations. For $k = o$, the topological group $C^o(M,S^1)$ carries the topology of compact convergence. In addition, $C^k(M,S^1)$ is a Banach manifold for each finite k and is a Fréchet manifold for $k = \infty$. (cf. [Go,Gui] p.76). However, let us describe a canonical chart of the unit element $\underline{1}$. Consider in $S^1 \times \mathbb{R}$ (the tangent bundle of S^1), the neighborhood $S^1 \times (-1,1)$ of $S^1\times\{o\}$. The set $\Phi(U)$ of all functions $f \in C^k(M)$ for which $(\underline{1},f)(M) \subset S^1\times(-1,1)$ forms an <u>open</u> <u>convex</u> subset of $C^k(M)$. On the other hand, the set U defined by $\{\varkappa\circ f \mid f \in \Phi(U)\}$ is open in $C^k(M,S^1)$ and $\Phi^{-1}:\Phi(U) \longrightarrow U$ assigning to each $f \in \Phi(U)$ the map $\varkappa\circ f \in U$ is a homeomorphism. Observe, now, that $\Phi(U)$ is a subspace of $\varkappa_M C^k(M)$. In fact $\Phi(U) \subset \varkappa_M C^k(M)$ is evenly covered in $C^k(M)$, the universal covering of $\varkappa_M C^k(M)$ as remarked in § 4.1. Since $\Phi(U)$ is a subspace of both $C^k(M,S^1)$ and $\varkappa_M C^k(M)$, the topological group $\varkappa_M C^k(M)$ is an open topological subgroup of $C^k(M,S^1)$. It is even the connected component W of $\underline{1} \in C^k(M,S^1)$. This can be seen as follows. As a manifold, modeled on convex charts, W is pathwise connected. Any path $\sigma:[-1,1] \to W$ in W starting at $\underline{1}$ and ending at t defines a homotopy $\sigma:[-1,1] \times M \to S^1$, connecting $\underline{1}$ and t. Without loss of generality we may assume that $t(p_o)=1$. Thus both $\underline{1}$ and $\underline{t}$ define the trivial homomorphism from the fundamental group $\Pi_1(M,p_o)$ of M into $\Pi_1(S,1)$ the fundamental group of S^1. But this means that $t \in \varkappa_M C^k(M)$. (In case of $C^o(M,S^1)$, we used the compactness of M only but not the differentiable structure; no arcwise connectedness is needed either, compare § 2.3)

Let us form the quotient $C^k(M,S^1)/\varkappa_M C^k(M)$, whose quotient structure is the discrete topology.

We have the following

<u>Lemma 10</u> For any $k = o,1\ldots,\infty$, the group $C^k(M,S^1)$ is dense in $C^o(M,S^1)$. Hence the inclusion $C^k(M,S^1) \subset C^o(M,S^1)$ induces an isomorphism $B:C^k(M,S^1)/\varkappa_M C^k(M) \longrightarrow C^o(M,S^1)/\varkappa_M C^o(M)$.

<u>Proof:</u> Consider $t \in C^o(M,S^1)$ and a map $(t,f) : M \to S^1 \times \mathbb{R}$ in the canonical chart of t, where $f \in C^o(M)$ composed with $\varkappa:\mathbb{R} \to S^1$ yields t. The set $C^\infty(M)$ is dense in $C^o(M)$. Hence we find a map $f_t \in C^\infty(M)$ close to f . But then $\varkappa \circ f_t$ is close to t, which proves the first assertion of the lemma. The second is a simple consequence.

As mentioned in § 2.3, we denote the quotient $C^o(M,S^1)/\varkappa_M C^o(M)$ by $\Pi^1(M)$. For any k, consider the homomorphism $b:C^k(M,S^1) \to \Pi^1(M)$. which is the canonical projection onto $C^k(M,S^1)/\varkappa_M C^k(M)$ followed by B. We now collect some of the material developed in this section:

<u>Proposition 11</u> For any $k=o,\ldots,\infty$, the sequence

$$\underline{1} \longrightarrow \varkappa_M C^k(M) \xrightarrow{i} C^k(M,S^1) \xrightarrow{b} \Pi^1(M) \longrightarrow 1 \quad ,$$

in which i denotes the inclusion map and in which $\Pi^1(M)$ carries the discrete topology, is a topological exact sequence.

<u>5.2. The character group of $C^k(M,S^1)$</u>

Proposition 11 yields immediately that

$$\underline{1} \longrightarrow \Gamma_c \Pi^1(M) \xrightarrow{*b} \Gamma_c C^k(M,S^1) \xrightarrow{*i} \Gamma_c \varkappa_M C^k(M) \quad \text{is exact.}$$

The maps $*b$ and $*i$ are defined by composing characters with b and i respectively. Moreover, $*i$ is surjective as we will later show. The techniques involved are based on the universal covering $\tilde{M}$ of M and the subsequent lemmas. For their formulation, let us introduce $u:\tilde{M} \longrightarrow M$, the covering map of M and, for any k, its induced maps

$$u^* : C^k(M) \longrightarrow C^k(\tilde{M}),$$

which is a homeomorphism onto a subspace of $C^k(\tilde{M})$,

$$u^{**} : L_c C^k(\tilde{M}) \longrightarrow L_c C^k(M) \quad \text{and finally ,}$$

$$*u^* : \Gamma_c C^k(\tilde{M}) \longrightarrow \Gamma_c C^k(M) \ .$$

These maps are defined by composing functions in $C^k(M)$ with u, linear maps in $L_c C^k(\tilde{M})$ with u^* and characters in $\Gamma_c C^k(\tilde{M})$ with u^* respectively. By the Hahn-Banach theorem, u^{**} and hence $*u^*$ are surjective.

A convergence vector space E and a convergence group G will be called L_c-embeddable and P_c-embeddable if i_E and j_E are homeomorphisms onto subspaces of $L_c L_c E$ and $\Gamma_c \Gamma_c G$ respectively.

<u>Lemma 12</u> For any $k=o,\ldots,\infty$ both $L_c C^k(\tilde{M}) \xrightarrow{u^{**}} L_c C^k(M)$ and $\Gamma_c C^k(\tilde{M}) \xrightarrow{*u^*} \Gamma_c C^k(M)$ are quotient maps in the categories of L_c-embeddable convergence vector spaces and Γ_c-embeddable convergence groups respectively.

<u>Proof:</u> First let us prove the second assertion. Let G be a Γ_c-embeddable group and $h : \Gamma_c C^k(\tilde{M}) \longrightarrow G$ a continuous homomorphism which factors over $*u*$ to $\bar{h} : \Gamma_c C^k(M) \longrightarrow G$. The canonical maps $\varkappa_* \circ i^k_M$ and $\varkappa_* \circ i^k_{\tilde{M}}$ from M into $\Gamma_c C^k(M)$ and from $\tilde{M}$ into $\Gamma_c C^k(\tilde{M})$ are again denoted by the symbols j^k_M and $j^k_{\tilde{M}}$ respectively. For any $\gamma \in \Gamma G$ the map $\gamma \circ h \circ j^k_{\tilde{M}}$ is of class C^k and factors over u to the C^k-function $\gamma \circ \bar{h} \circ j^k_M$. Hence $*\bar{h}(\gamma) = \gamma \circ \bar{h} \in \Gamma_c \Gamma_c C^k(M)$. Since $*h = **u*\ *\bar{h}$, the map $*\bar{h}$ is continuous, and since G is P_c-embeddable h is continuous. The first assertion is verified analogously.

The next lemma employs $*u : C^k(M,S^1) \longrightarrow C^k(\tilde{M},S^1)$, defined by $t \circ u$ for each $t \in C^k(M,S^1)$. Since $\tilde{M}$ is simply connected, we have $C^k(\tilde{M},S^1) = \varkappa_{\tilde{M}} C^k(\tilde{M})$ for any k. Restricting $*u$ to $\varkappa_M C^k(M)$, we obtain the continuous homomorphism $**u : \Gamma_c \varkappa_M C^k(M) \longrightarrow \Gamma_c \varkappa_{\tilde{M}} C^k(\tilde{M})$, defined by composing the characters with $*u | \varkappa_M C^k(M)$.

<u>Lemma 13</u> The homomorphism $\Gamma_c \varkappa_{\tilde{M}} C^k(\tilde{M}) \xrightarrow{\ **u\ } \Gamma_c \varkappa_M C^k(M)$ is a quotient map in the category of P_c-embeddable convergence groups.

<u>Proof:</u> In order to prove surjectivity, let us consider

$$\begin{array}{ccc} \Gamma_c \varkappa_M C^k(\tilde{M}) & \xrightarrow{\ *\varkappa_{\tilde{M}}\ } & \Gamma_c C^k(\tilde{M}) \\ \downarrow{\scriptstyle **u} & & \downarrow{\scriptstyle *u*} \\ \Gamma_c \varkappa_M C^k(M) & \xrightarrow{\ *\varkappa_M\ } & \Gamma_c C^k(M) \end{array} ,$$

for each $k = o,\ldots,\infty$, where $*\varkappa_M$ and $*\varkappa_{\tilde{M}}$ are defined in the usual way, namely by composing the character with $\varkappa_M$ and $\varkappa_{\tilde{M}}$ respectively. We first show that $**u$ is surjective. Consider $\gamma \in \Gamma_c \varkappa_M C^k(M)$ and form $*\varkappa_M(\gamma)$. By lemma 12 we can find a $\bar{\gamma} \in \Gamma_c C^k(\tilde{M})$ with $\bar{\gamma} \circ u* = *\varkappa_M(\gamma)$. Since $*\varkappa_M(\gamma)(\underline{\mathbb{Z}}) = \underline{1}$, and since $u*|\mathbb{Z} = id_{\mathbb{Z}}$, we have $\bar{\gamma}(\underline{\mathbb{Z}}) = \underline{1}$. Hence we find $\gamma^1 \in \Gamma_c \varkappa_M C^k(M)$ with $\gamma^1 \circ \varkappa_M = \bar{\gamma}$. Since $*\varkappa_M$ and $*\varkappa_{\tilde{M}}$ are injective, we have $**u(\gamma^1) = \gamma$. To verify the rest of the lemma, one proceeds analogously as in the proof of lemma 12, or one uses lemma 12 in connection with the direct decomposition of theorem 6.

We collect some of our results on the character group of $C^k(M,S^1)$ in the following theorem.

<u>Lemma 14</u> For any $k=0,\dots,\infty$ the topological exact sequences

$$\underline{0} \longrightarrow \underline{\mathbb{Z}} \xrightarrow{i_1} C^k(M) \xrightarrow{\varkappa_M} \varkappa_M C^k(M) \longrightarrow \underline{1}$$

and

$$\underline{1} \longrightarrow \varkappa_M C^k(M) \xrightarrow{i} C^k(M,S^1) \xrightarrow{b} \Pi^1(M) \longrightarrow 1$$

have exact duals, namely

$$\underline{1} \longrightarrow \Gamma_c \varkappa_M C^k(M) \xrightarrow{*\varkappa_M} \Gamma_c C^k(M) \xrightarrow{*i_1} \Gamma_c \underline{\mathbb{Z}} \cong S^1 \longrightarrow 1 ,$$

and

$$\underline{1} \longrightarrow \Gamma_c \Pi^1(M) \xrightarrow{*b} \Gamma_c C^k(M,S^1) \xrightarrow{*i} \Gamma_c \varkappa_M C^k(M) \longrightarrow \underline{1} .$$

Here $*\varkappa_M$ and $*b$ are homeomorphisms onto their ranges, $*i_1$ is a quotient map and $*i$ is a quotient map in the category of P_c-embeddable groups.

<u>Proof:</u> Since $\varkappa_M$ is a quotient map

$$\underline{1} \longrightarrow \Gamma \varkappa_M C^k(M) \xrightarrow{*\varkappa_M} \Gamma_c C^k(M) \xrightarrow{*i_1} \Gamma_c \mathbb{Z}$$

is right exact and $*\varkappa_M$ is a bicontinuous isomorphism onto a subspace. To show that the last map $*i_1$, which is a restriction map, is surjective, we extend a given character $\gamma \in \Gamma_c \mathbb{Z}$ onto $\mathbb{R}$, turn it via $\varkappa_*$ into a real-valued functional ℓ and extend this functional ℓ to $\ell' \in L_c C^k(M)$. Obviously, $\varkappa \circ \ell' = \gamma$. To show that $*i_1$, is a quotient map, one proceeds as in [Bi,2], p.71. To demonstrate that the second dual sequence is exact, we point out that the sequence

$$\underline{1} \longrightarrow \Gamma_c \Pi^1(M) \xrightarrow{*b} \Gamma_c C^k(M,S) \xrightarrow{*i} \Gamma_c \varkappa_M C^k(M)$$

is right exact, where $*b$ maps its domaine homeomorphically onto its range, regarded as a subspace of $\Gamma_c C^k(M,S^1)$.

To verify the surjectivity of $*i$, we form the commutative diagram:

$$\begin{array}{ccc} \Gamma_c \varkappa_{\tilde{M}} C^k(\tilde{M}) & \xrightarrow{\;**u\;} & \Gamma_c C^k(M,S^1) \\ \Big\downarrow{\scriptstyle **u} & \swarrow{\scriptstyle *i} & \\ \Gamma_c \varkappa_M C^k(M) & & \end{array}$$

From this, we conclude via lemma 13, the last part of the above theorem.

5.3. The P_c-reflexivity of $C^k(M,S^1)$

For any $k=o,\ldots,\infty$ consider the commuting diagram

$$\begin{array}{ccccccccc} \underline{1} & \longrightarrow & \Gamma_c\Gamma_c\varkappa_M C^k(M) & \xrightarrow{**i} & \Gamma_c\Gamma_c C^k(M,S^1) & \xrightarrow{**b} & \Gamma_c\Gamma_c \Pi^1(M) & \longrightarrow & \underline{1} \\ & & \Big\uparrow{\scriptstyle j_{\varkappa_M C^k(M)}} & & \Big\uparrow{\scriptstyle j_{C^k(M,S^1)}} & & \Big\uparrow{\scriptstyle j_{\Pi^1(M)}} & & \\ \underline{1} & \longrightarrow & \varkappa_M C^k(M) & \xrightarrow{i} & C^k(M,S^1) & \xrightarrow{b} & \Pi^1(M) & \longrightarrow & 1 \end{array} ,$$

where $**i$ and $**b$ are defined by composing the respective characters with $*i$ and $*b$.

Both the discrete topological group $\Pi^1(M)$ and $\varkappa_M C^k(M)$ are P_c-reflexive (theorem 8). Using lemma 14, one easily verifies the exactness of the upper sequence. By the five lemma, $j_{C^k(M,S^1)}$ has to be an isomorphism. Evidently $j_{C^k(M,S^1)}$ is continuous. To see that its inverse is continuous we form $j_M^k : M \longrightarrow \Gamma_c C^k(M,S^1)$, defined by $j_M^k(p)(t)=t(p)$ for all $p \in M$ and all $t \in C^k(M,S^1)$.
The dual map

$$*j_M^k : \Gamma_c\Gamma_c C^k(M,S^1) \longrightarrow C^k(M,S^1) ,$$

sends each $\gamma \in \Gamma_c\Gamma_c C^k(M,S^1)$ into $*j_M^k(\gamma) = \gamma \circ j_M^k$. Since $*j_M^k \circ j_{C^k(M,S^1)} = id_{C^k(M,S^1)}$, we obtain the continuity of $j^{-1}_{C^k(M,S^1)}$.

Therefore we may conclude with:

<u>Theorem 15</u> For any $k=o,\ldots,\infty$ the topological group $C^k(M,S^1)$ is P_c-reflexive.

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