APPLICATIONS OF THE SIDE APPROXIMATION THEOREM FOR SURFACES

R. H. BING

Madison

By a surface we mean a closed set in a 3-manifold which is locally like the plane – each point of the surface lies in an open subset of the surface which is topologically equivalent to the interior of a circle.

Two surfaces may intersect in a peculiar set in Euclidean 3-space. Even two smooth surfaces may intersect in a set locally like the closure of the graph of $y = \sin(1/x)$. In fact, if K is any closed subset of a round 2-sphere S, there is a second two sphere S' (not necessarily round) such that S' contains K and lies except for K on the interior of S. On the other hand, the intersection of two polyhedral surfaces is always a polyhedron. (A surface is polyhedral if it is a geometric complex – the sum of a locally finite collection of triangular planar disks.) To bring about a reasonable intersection of surfaces, the Approximation Theorem for surfaces was proved. It is proved in [2] and may be stated as follows.

Theorem 1. Approximation Theorem for Surfaces. *Suppose S is a surface in a triangulated 3-manifold M and f is a non negative continuous function defined on S. Then there is a surface S' and a homeomorphism h of S onto S' such that*

a) $p(x, h(x)) \leqq f(x),\ x \in S$ *and*

b) S' *is locally polyhedral at* $h(x)$ *if* $f(x) > 0$.

This Approximation Theorem for Surfaces has had several applications. O. G. Harrold used it [10] to show that if P is polygon in E^3 which bounds a disk, then P bounds a polyhedral disk. He also used it to show that in various theorems where the hypothesis has previously required that certain 2-spheres be locally polyhedral on certain parts, these extra conditions could be dropped.

E. E. Moise has shown [11] that if h is a homeomorphism of an open subset U of E^3 into E^3 and f is a non negative continuous function defined on U, then there is a homeomorphism h' of U onto $h(U)$ such that

a) h' is locally polyhedral at $x \in U$ if $f(x) > 0$ and

b) $p(h(x), h'(x)) \leqq f(x)$.

This is a very useful theorem and was used in [12] by Moise to prove that any 3-manifold can be triangulated.

The Approximation Theorem for Surfaces was extended to an Approximation

Theorem for 2-complexes in [6] to give alternative proofs of these two results of MOISE.

In hearing papers given about mappings on 3-manifolds, I frequently hear the restrictions, "We restrict ourselves to piecewise linear homeomorphisms". While this restriction is sometimes necessary (as in the case of certain periodic homeomorphisms) it frequently is not and shows that the author is not proving as strong a theorem as he might have proved if he were acquainted with the results mentioned in the preceding two paragraphs.

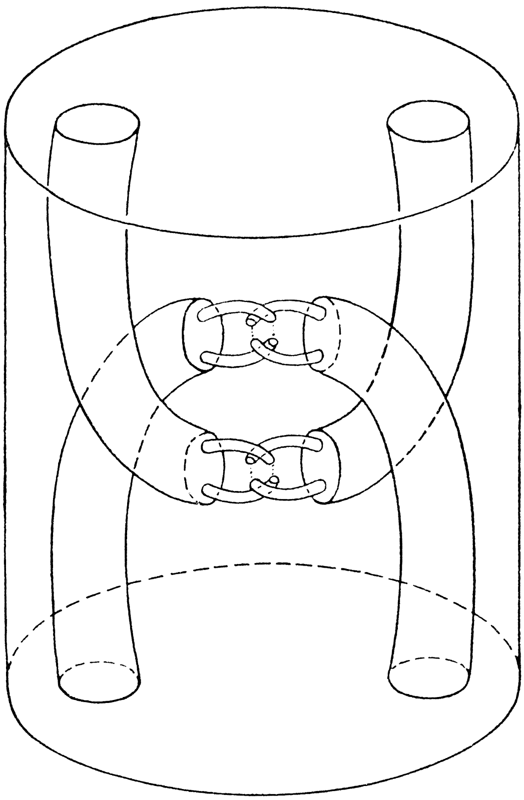

Fig. 1.

If S is a 2-sphere and T is a tetrahedron in E^3, there is a homeomorphism of S onto BdT (the boundary of T). If this homeomorphism can be extended to take E^3 onto itself, we say that S is tame. If the homeomorphism cannot be extended, we say it is wild. The 2-sphere shown in Figure 1 was obtained by starting with the boundary of a cylindrical can, replacing disks in the ends of the can by tubes that almost link, replacing disk in the ends of tubes with tubes, etc.

It is a wild 2-sphere since its interior (bounded complementary domain) is not simply connected (1-connected). Note that the 2-sphere can be approximated by a polyhedral 2-sphere which lies on the exterior of the wild 2-sphere. However, it cannot be approximated from the interior of the wild 2-sphere since an approximating polyhedral 2-sphere which was on the interior would cut off some of the small tubes. Hence, at best some parts of the approximating 2-sphere would fail to lie on the interior of the wild 2-sphere.

In a seminar BOB WILLIAMS asked if it might be possible to get a polyhedral approximation to any 2-sphere which would lie except for small disks on a prescribed side of the 2-sphere. It turns out that the answer is in the affirmative as given by the following result.

Theorem 2. Side Approximation Theorem for 2-Spheres. *Suppose S is a 2-sphere in E^3, $\varepsilon > 0$, and U is a component of $E^3 - S$. Then there is a polyhedral 2-sphere S' and a homeomorphism h of S onto S' such that*

a) *h moves no point more than ε and*

b) *S' contains a finite family of mutually exclusive disks, each of diameter less than ε, such that S' minus these disks lies in U.*

The proof of this theorem has been distributed in ditto form and will appear in the Annals of Mathematics [3]. The proof is modeled (with certain simplifications) after the proof the Approximation Theorem for Surfaces given in [2]. The construction is as might be expected in that S is used as a model and a polyhedral surface S' is constructed beside S with S' not being permitted to intersect S except in some feelers.

Without using the full strength of the Side Approximation Theorem but using merely the Approximation Theorem for Surfaces the following result had been proved [4]:

Theorem 3. *A 2-sphere S in E^3 is tame if for each $\varepsilon > 0$ and each component U of $E^3 - S$ there is a 2-sphere S in U, and a homeomorphism of S onto S' that moves no point more than ε.*

By using this result and the Side Approximation Theorem the following result may be established:

Theorem 4. *A 2-sphere S in E^3 is tame if $E^3 - S$ is 1-ulc.*

We say that a set is 1-*ulc* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if D is a disk and f is a map of BdD into a δ-subset of X, then f can be extended to map D into an ε-subset of X.

Outline of proof of Theorem 4. Theorem 4 is proved in [5] but the outline of the proof is as follows. We shall apply Theorem 3 so we only need to show that for each component U of $E^3 - S$, there is a 2-sphere in U that is a close homeomorphic approximation to S. It follows from the Side Approximation Theorem that there is a polyhedral 2-sphere S' which lies except for small polyhedral disks $D_1, D_2, \ldots, D_n$ in U. Since U is 1-*ulc*, D_i can be shrunk to a point on a small polyhedral singular disk E_i in U. Then $(S' - \sum D_i) + \sum E_i$ is a singular 2-sphere in U. By using results of C. D. Papakyriakopoulos regarding Dehn's lemma [3] the singular 2-sphere may be changed to a 2-sphere in U which is a close approximation to S.

Theorem 4 may be extended to apply to surfaces as well as 2-spheres. How this may be done and the details of the proof of Theorem 4 are to be found in [5].

A Sierpinski curve is a 1-dimensional subset of a disk D obtained by deleting from D a null sequence of interiors of disks such that each of these small disks lies on the interior of D and no two of them intersect each other. We say that a Sierpinski curve is tame if it lies on a tame 2-sphere. The following is another application of the Side Approximation Theorem which gives a strong affirmative answer to the question as to whether or not each 2-sphere in E^3 contains a tame arc.

Theorem 5. *For each 2-sphere S in E^3 and each $\varepsilon > 0$ there is a tame Sierpinski curve X on S such that each component of $S - X$ is of diameter less than ε.*

Outline of proof. It follows from an extension of Theorem 2 as shown in Figure 2 that there is a polyhedral 2-sphere S' such that S' lies except for small feelers (like F_1) on Int S. Cut off feelers like F_1 and replace them with a smooth disk so that the resulting 2-sphere S_1 lies on Int S'. An approximation to S_1 almost from the other side enables us to replace F_2 by a smooth disk so that the resulting 2-sphere S_2 can be approximated closely from both sides. In a similar fashion we replace S_2 by S_3 so as to remove small feelers like F_3 and replace S_3 by S_4 to remove feelers like F_4. If great care is exercised as done in [8], S_1, $S_2, \ldots$ converges to a 2-sphere S_∞ which shares a Sierpinski curve X with S so that each component of $S - X$ is of diameter less than ε. Furthermore S_∞ was chosen so that it could be homeomorphically approximated arbitrarily closely from both sides. It follows from Theorem 3 that S_∞ is tame.

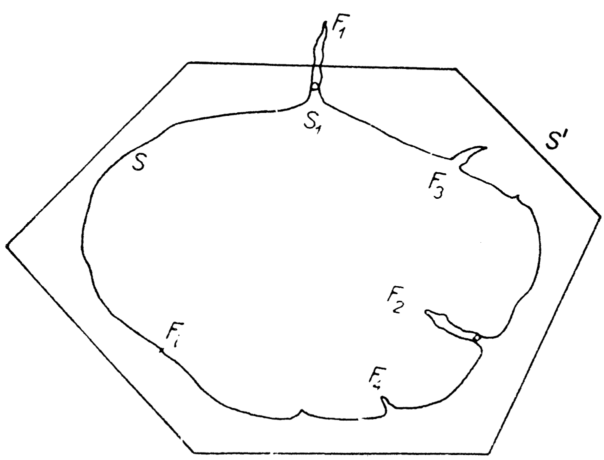

Fig. 2.

Some peculiar wild 2-spheres have been described. Some have the property that although their complements are simply connected, no disk in them is tame but each arc in them is tame. Some of them are even such that each Sierpinski curve in them is tame. David Gillman and W. R. Alford have obtained such examples as modification of the ones given in [7]. However, Theorem 5 reveals that any 2-sphere whatsoever in E^3 can be obtained from a tame 2-sphere by removing small disks and putting on feelers. Also, the reversal of this operation changes every 2-sphere into a tame 2-sphere. This adds to our understanding of wild 2-spheres. Alford has shown [1] that there are uncountably many different kinds of wild 2-spheres.

Theorem 5 is used to prove the following result:

Theorem 6. *Each 2-sphere in each 3-manifold can be pierced by a tame arc.*

Details of the proof of the above result are found in [9] where it is shown that the 2-sphere is pierced at each point of the tame Sierpinski curve. D. Gillman has extended Theorem 6 to show that the places at which the 2-sphere can be pierced by a tame arc are actually the points of the 2-sphere which lie on tame arcs in the 2-sphere.

The following questions suggest themselves:

Question 1. What is the category of the points of a 2-sphere in E^3 at which it cannot be pierced by a tame arc? Conjecture – a 0-dimensional F_σ set.

Question 2. Can Dehn's lemma be extended to imply that if f is a map of a disk D into E^3 such that f^{-1} is $1-1$ on $f(\mathrm{Bd}D)$, then $f(\mathrm{Bd}D)$ bounds a disk in E^3.

Question 3. Can a homotopy 3-cell in E^3 be approximated by a real cell in the following sense.

Suppose U is a bounded, contractible open subset of E^3 and V is a neighborhood of $\mathrm{Bd}U$. Is there a 3 cell C in E^3 such that $U - V \subset C \subset U + V$.

Question 4. (Raised by Borsuk). If X is a contractible continuous curve in E^3 and $\varepsilon > 0$, is there a map f of X onto a contractible polyhedron such that f moves no point more than ε?

References

[1] *W. R. Alford*: Some „nice" wild two spheres in three space. Notices Amer. Math. Soc., vol. *8* (1961), 348.

[2] *R. H. Bing*: Approximating surfaces with polyhedral ones. Ann. Math., vol. *65* (1957), 456–483.

[3] *R. H. Bing*: Approximating surfaces from the side. Ann. Math. to appear.

[4] *R. H. Bing*: Conditions under which a surface in E^3 is tame. Fund. Math., vol. *47* (1959), 105–139.

[5] *R. H. Bing*: A surface is tame if its complement is 1-*ulc*. Trans. Amer. Math. Soc., vol. *101* (1961), 294—305.

[6] *R. H. Bing*: An alternative proof that 3-manifolds can be triangulated. Ann. Math., vol. *69* (1959), 37–65.

[7] *R. H. Bing*: A wild surface each of whose arcs is tame. Duke Journal, *28* (1961), 1–15.

[8] *R. H. Bing*: Each disk in E^3 contains a tame arc, Am. J. Math., to appear.

[9] *R. H. Bing*: Each disk in E^3 is pierced by a tame arc, Am. J. Math., to appear.

[10] *O. G. Harrold Jr.*: Some consequences of the approximation theorem of Bing. Proc. Amer. Math. Soc., vol. *8* (1961), 204–206.

[11] *Edwin E. Moise*: Affine structures in 3-manifolds, IV, Piecewise linear approximations of homeomorphisms. Ann. Math., vol. *66* (1959), 215–222.

[12] *Edwin E. Moise*: Affine structures in 3-manifolds, V, The triangulation theorem and Hauptvermutung. Ann. Math., vol. *56* (1952), 96–114.

[13] *C. D. Papakyriakopoulos*: On Dehn's lemma and the asphericity of knots. Ann. Math., vol. *66* (1951), 1–26.