CONCERNING INFINITE-DIMENSIONAL SPACES

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The problem whether every infinite-dimensional compactum (= compact metric space) contains closed subsets of an arbitrary finite dimension, was formulated by myself some 35 years ago and it still remains open (even for closed one-dimensional subsets).

In this note some theorems concerning this problem are considered.

A metric space of infinite dimension is called countable-dimensional if it is a union of a countable number of 0-dimensional subsets. In the opposite case we call the space strongly infinite-dimensional.

The following definitions generalize a classical notion due to Urysohn.

An infinite-dimensional compactum is called a Cantorian manifold in the weak sense (or in the strong sense, respectively), if it cannot be decomposed by any finite-dimensional (or by any finite- or countable-dimensional, respectively) closed subset.[1])

The proof of the following theorem 1 is very easy:

Theorem 1. *An infinite-dimensional compactum X contains closed subset of an arbitrary finite dimension if and only if it contains some countable-dimensional closed set.*

Theorem 2 improves my older result [1].

Theorem 2. *Let X be an infinite-dimensional compactum. Then either*

a) *X contains a countable-dimensional closed set*

or

b) *X contains an infinite-dimensional Cantorian manifold in the strong sense.*

The two cases do not exclude each other.

However, the question whether every infinite-dimensional Cantorian manifold in the strong sense contains a countable-dimensional closed subset, still remains open.

Now we shall consider arbitrary separable metric spaces.

Theorem 3. *Under the assumption of the continuum hypothesis every strongly infinite-dimensional separable metric space X contains a set A with the following property:*

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[1]) In the weak case we suppose moreover that the space can be decomposed by some countable-dimensional closed subset.

The intersection of A with every finite-dimensional or countable-dimensional subset of X is at most countable.

This theorem generalizes a result of W. HUREWICZ [2]. The proof makes use (as does the construction by Hurewicz) of the fact (proved by myself in the year 1925) that every n-dimensional subset of a separable metric space X (with a given metric) is contained in some G_δ-set of the same dimension, lying in the metric space X.

Concerning the countable dimensional spaces, I have proved in [3] the

Theorem 4. *Every countable-dimensional separable metric space X is a union*

$$X = \bigcup_{i=1}^{\infty} \mathfrak{M}_i$$

of 0-*dimensional subsets* $\mathfrak{M}_i$ *such that the sum of any finite number of them is still* 0-*dimensional*:

$$\dim \bigcup_{i=1}^{N} \mathfrak{M}_i = 0 \text{ for any finite } N .$$

Let us finally point out that even the question whether in every countable dimensional separable metric X there is contained a subset of an arbitrary fiinite dimension, still remains open.

References

[1] *Л. А. Тумаркин:* О бесконечномерных канторовых многообразиях. Докл. АН СССР, *115* (1957), 244—246.

[2] *W. Hurewicz*: Une remarque sur l'hypothèse du continu. Fundam. Math., *19* (1932), 8–16.

[3] *Л. А. Тумаркин:* О разбиении пространств на счетное число нульмерных множеств. Вестн. Моск. ун-та, серия I, математика, механика, № 1 (1960), 25—33.