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LINEARIZATION OF MAPPINGS

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A brief discussion of two theorems in this area.

Let M be a metrizable space and G a compact topological transformation group of homeomorphisms of M onto M. It is clear when such a pair G, M is called (topologically) equivalent to a pair G^*, M^*.

Theorem I. *To every pair G, M there corresponds an equivalent pair G^*, M^* where M^* is embedded in some suitable real Hilbert space H^* and the action of G^* on M^* can be extended over all of H^* in such a way that G^* acts as a (compact) group of unitary homeomorphisms of H^* onto H^*.*

Briefly: *the action of G on M is linearized by a group of unitary transformations in Hilbert space.*

Sketch of the proof. M may be thought of as being embedded into a bounded subset of some real Hilbert space H. Introduce an orthogonal coordinate system in H, and a point $x \in H$ will have coordinates $(x)_\alpha$, α running through some index-set A. We define for every $x \in H$ a map

$$\tau : x = (x)_\alpha \to x^* = (gx)_\alpha\,, \quad \alpha \in A\,, \quad g \in G\,,$$

where gx is the image of x under g in M and $(gx)_\alpha$ is thought of as a functional depending on the two variables g and α. Observe that τ is one-one. We will embed the set $\{x^*\} = M^*$ into a Hilbert space H^*. In order to define H^* we proceed as follows.

The vector space $\mathbf{V}$ will consist of all finite linear combinations of points x^* over the real field, where addition and multiplication with a real scalar are defined in the natural way. For two such vectors $\mathbf{v}$ and $\mathbf{w}$

$$\mathbf{v} = \sum_{j=1}^{n} a_j(gx_j)_\alpha\,, \quad \mathbf{w} = \sum_{j=1}^{n} b_j(gy_j)_\alpha$$

we define an inner product

$$(\mathbf{v}, \mathbf{w}) = \int_G \sum_\alpha (\mathbf{v}\,.\,\mathbf{w})\,\mathrm{d}g\,.$$

Observe that this makes sense. Thus the vector space $\mathbf{V}$ becomes an, in general, still incomplete Hilbert space. Its completion will be H^*. One can prove that τ is a topological map of M onto M^*, while the action of

$$G^* = \tau G \tau^{-1} \quad \text{on} \quad M^*$$

is defined in a natural way over all of H^*. The invariance of integration shows that G^* acts in this way as a group of unitary transformations.

Two unsolved problems:

1° If G is locally compact, can we find a G^* of bounded linear operators?

2° The same question, if G is a compact semigroup of continuous mappings of M into itself.

If P is a topological product of an infinite number, say m copies of one and the same topological space T, every permutation of these m copies induces in a natural way, an autohomeomorphism of P.

In the same way every *immutation* (an immutation is defined as a map of a set – in our case of power m – into itself) defines, in the natural way, a continuous map of P into itself. If T is a vector space, such an immutation is a linear map.

A family of immutations of a given set generates a semigroup of immutations. Conversely, if some semigroup is given, we may add a unit element to the semigroup. The set of all left (or right) multiplications carried out on the elements of the latter semigroup defines a semigroup of immutations (on the set of elements of the semigroup) which is isomorphic (anti-isomorphic) to the latter semigroup. In particular a free semigroup F of power m with identity element may be represented isomorphically by the corresponding free immutation semigroup of left immutations.

In the sequel let P be a product of m segments. The free semigroup F may be represented as a set of immutations of the m segments, inducing a free semigroup of continuous maps of P into itself. We might call these maps "linear" (since we can extend the segments to real lines). This defines the pair F, P.

Take a set of free generators φ of F. How does such a φ look like as immutation, i.e. as coordinate transformation on the m coordinates x_α of P? For every such φ there corresponds a splitting of the m coordinate-indices α into m countable sets of indices β, i, where β is an index-set of power m and $i = 1, 2, 3, \ldots$ ad inf. The corresponding coordinate transformation induced by φ is given by

$$(*) \qquad y_{\beta,i} = x_{\beta,i+1} \quad \text{for all pairs } \beta, i\,.$$

Every completely regular space R of weight $\leq m$ admits a topological embedding into P. We might say P is a universal space regarding the family of spaces R. Now let, moreover, be given a set S of m arbitrary continuous mappings of R into itself. Without loss of generality we may assume that S is a semigroup with identity element. This defines the pair S, R.

Theorem II. *Any pair S, R admits a universal linearization by means of the pair F, P.*

Explicitly: it is possible to embed R in such a way into P, that the action of F onto P, restricted to the embedded R, coincides with the action of S onto the embedded R. So, in particular the action of any $s \in S$ on the embedded R can be extended over all of P.

Every such an extension map is a "linear" map of type (*).

Remarks. Analogous results hold for sets S of power different from m. The action of F on the embedded R is not effective, in general. A corresponding theorem holds for autohomeomorphism groups S. In this case F is a free group and i runs through all integers in the equations (*).

Indication of proof. Set up a one to one correspondence

$$\varphi \leftrightarrow s \quad (s \neq e)$$

between the free generators of F and the elements $(\neq e)$ of S. This correspondence induces a homomorphic map ω of F onto S.

The elements of F will be denoted by ψ and F will also serve as an index-set. We can write

$$P = \prod_{\substack{\lambda \in L \\ \psi \in F}} I_{\lambda,\psi} ,$$

where $L$is an index set of power m and every $I_{\lambda,\psi}$ is a segment. A point $x \in P$ has coordinates $x_{\lambda,\psi}$

$$x = (x_{\lambda,\psi})_{\substack{\lambda \in L \\ \psi \in F}} \quad (0 \leqq x_{\lambda,\psi} \leqq 1) .$$

For every fixed element $\gamma \in F$ we determine a "linear" map γ of P into itself by the following immutation ($x\gamma$ denotes the image of x under γ)

$$x\gamma = (x\gamma_{\lambda,\psi})_{\substack{\lambda \in L \\ \psi \in F}} \overset{\text{def}}{=} (x_{\lambda,\gamma\psi})_{\substack{\lambda \in L \\ \psi \in F}} .$$

Furthermore, one may think R to be contained topologically (this is a preliminary embedding) in the subspace of P spanned by the segments $I_{\lambda,\varepsilon}$ ($\lambda \in L$, ε = identity-index of F). So a point $y \in R$ has coordinates

$$y = (y_{\lambda,\psi})$$

with

$$y_{\lambda,\psi} = 0 \quad \text{if} \quad \psi \neq \varepsilon , \quad \psi \in F .$$

The final embedding $R^* = \{y^*\}$ of R is determined by a map τ of R into P:

$$\tau \colon\ y \to y^* = (y^*_{\lambda,\psi})_{\substack{\lambda \in L \\ \psi \in F}} \overset{\text{def}}{=} (y\ \omega(\psi)_{\lambda,\varepsilon})_{\substack{\lambda \in L \\ \psi \in F}} , \tag{1}$$

where $y\ \omega(\psi)$ is the image of y under $s = \omega(\psi)$, so a point of R in its first embedding.

One can show that τ is a homeomorphism, while moreover the action of an element $\gamma \in F$ on R^* coincides with the action of $\omega(\gamma)$ on R. Moreover, it appears that the requirements of theorem II are fulfilled.

Some other results:

J. de Groot: Every continuous mapping is linear. Notices Amer. Math. Soc. *6* (1959), 754.

A. H. Copeland Jr. and *J. de Groot*: Linearization of a homeomorphism, Math. Ann. *144* (1961), 80–92.