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ON HOMOLOGY THEORY OF NON-CLOSED SETS

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1. Direct systems of compact groups. In homology theory of non-closed sets the approximation of sets by their compact subsets or by their neighbourhoods is of decisive importance. Such approximations lead, in particular, to direct systems of compact groups. The definition of the limit of such systems given below would seem to have proved of some use in homology theory of non-compact spaces [2b, c; 3; 6c, d, f].

Let $\{A_\alpha, \pi_{\alpha\beta}\}$ be a direct system of compact groups A_α with homomorphisms

$$\pi_{\alpha\beta} : A_\alpha \to A_\beta .$$

Let B_α be the character-group of A_α and $\sigma_{\beta\alpha} : B_\beta \to B_\alpha$ a homomorphism, satisfying the permanence relation

$$(a_\alpha, \sigma_{\beta\alpha} b_\beta) = (\pi_{\alpha\beta} a_\alpha, b_\beta), \quad a_\alpha \in A_\alpha , \quad b_\beta \in B_\beta .$$

Then $\{B_\alpha, \sigma_{\beta\alpha}\}$ is an inverse system of discrete groups, and $\{A_\alpha, \pi_{\alpha\beta}\}$ and $\{B_\alpha, \sigma_{\beta\alpha}\}$ are dually paired to the group κ of real numbers mod 1. Let B be the limit-group of $\{B_\alpha, \sigma_{\beta\alpha}\}$ with *discrete* topology, and A' the usual algebraic limit-group of $\{A_\alpha, \pi_{\alpha\beta}\}$. Then the groups A' and B are paired to κ under a multiplication defined as follows:

$$(a, b) = (a_\alpha, b_\alpha), \quad \text{where} \quad a_\alpha \in a \in A' , \quad b_\alpha \in b \in B .$$

Let A_0 be the annihilator of B in A'. We use the induced pairing of the factor-group A'/A_0 and the group B to introduce a topology in A'/A_0 as follows: for any finite subset F of B and any nucleus V of κ a nucleus U of A'/A_0 is defined as a set of elements a of A'/A_0 satisfying the condition $(a, F) \in V$. We call the group A'/A_0 in this topology the *general limit-group* of $\{A_\alpha, \pi_{\alpha\beta}\}$, and the compact completion A of A'/A_0, which exists and is unique, the *limit-group* of $\{A_\alpha, \pi_{\alpha\beta}\}$:

$$A = \varinjlim \{A_\alpha, \pi_{\alpha\beta}\} .$$

The group A'/A_0 may be topologically imbedded in the character-group of B as an everywhere dense subgroup of it and, therefore, A and B are dual: $A \mid B$.

Now we can introduce a compact topology in *the direct sum* $\sum A_\alpha$ *of compact groups* A_α. To do this, it is sufficient to consider $\sum A_\alpha$ as the limit in the above sense of the direct system of all groups A'_α with inclusion homomorphisms, where A'_α is the sum of a finite subsystem of the system $\{A_\alpha\}$.

Conversely, we can first define the compact sum $\sum A_\alpha$ of the compact groups A_α, and then the limit of the system $\{A_\alpha, \pi_{\alpha\beta}\}$. To do this, we consider the topology of the usual sum $\sum A_\alpha$ which satisfies the following conditions: a) the inclusion homomorphism

$$i_\alpha : A_\alpha \to \sum A_\alpha$$

is continuous for every α; b) every homomorphism f of $\sum A_\alpha$ in any compact group C is continuous, if the homomorphisms fi_α are continuous; c) in this topology $\sum A_\alpha$ has a compact completion. Such a topology of $\sum A_\alpha$ exists and is unique. The compact completion of $\sum A_\alpha$ will be called the *direct sum of compact groups* A_α and denoted by $\sum A_\alpha$. Now, the factor group $\sum A_\alpha / A_0$, where A_0 is the closure of the subgroup generated by the elements $a_\alpha - \pi_{\alpha\beta} a_\alpha$, is the *limit*

$$\varinjlim \{A_\alpha, \pi_{\alpha\beta}\} .$$

2. Projective and spectral groups of complexes and spaces. Let $\{K_\alpha, <\}$ be a directed system of all finite closed subcomplexes K_α of a complex K, ordered by the inclusion

$$\alpha < \beta \Leftrightarrow K_\alpha \subset K_\beta .$$

The groups of r-chains of K_α over a discrete or compact group of coefficients X, with the homomorphisms $i_{\alpha\beta *}$ induced by the inclusion maps

$$i_{\alpha\beta} : K_\alpha \to K_\beta ,$$

form a directed system of groups

$$\{C_r(K_\alpha, X), i_{\alpha\beta *}\} . \tag{1}$$

On the basis of (1), we construct homology groups of K of two kinds: projective and spectral. *Projective homology groups* are obtained if we first take the limit of the system (1) and then apply the homological functor, or, in notation,

$$H_r \varinjlim \{C_r(K_\alpha, X), i_{\alpha\beta *}\} ,$$

the limit being understood in the sense of § 1. Here, when X is compact, the boundary operator is first defined for the general limit-group and then extended by continuity to the limit-group. *Spectral homology groups* are obtained if, on the contrary, we first apply the homological functor and then take the limit; in notation,

$$\varinjlim \{H_r(K_\alpha, X), i_{\alpha\beta *}\} .$$

Using the inverse system of cochain groups of K_α over a discrete or compact group of coefficients Y

$$\{C^r(K_\alpha, Y), i^*_{\beta\alpha}\} \tag{2}$$

we obtain, similarly, the *projective* and *spectral cohomology groups* of K.

If X and Y are dual, then the projective [spectral] homology group of K over X and the projective [spectral] cohomology group of K over Y are dual not only for a discrete X, but also for a compact X.

From the *theorem on commutativity of limit operator and homology functor,*[1]) now a proposition of homological algebra, it follows that *projective and spectral homology groups are isomorphic for a discrete group of coefficients* X, while *projective and spectral cohomology groups are isomorphic for a compact group of coefficients* Y. Examples show that these isomorphisms are not valid when X is compact and Y discrete.

Making use of the spectral and projective groups of complexes we construct the corresponding (i. e. spectral and projective) homology and cohomology groups of spaces of various types, viz., singular, continuous, Čech, Vietoris, etc. In this way we obtain, on the one hand, the usual singular, Vietoris, etc. groups, every one of these groups being of either the spectral or the projective kind. On the other hand, we obtain new groups which are opposite in kind to the usual groups just mentioned. Moreover, the above definitions, especially that of the limit of direct system of compact groups, makes it possible to construct homology and cohomology groups of spaces not only over a discrete, but over a compact group of coefficients as well. As is known, these latter have not all been previously defined (see, e. g., [8], pp. 166, 184, 185, 188, 223, 233 and [10], p. 393).

Taking the singular complex of a space and its projective and spectral groups, we obtain, apart from the usual singular groups of the space – which are groups of the projective kind – also the *spectral singular groups*, as well as the *projective singular homology groups with compact coefficients.*

The *continuous* homology groups are discrete groups of the spectral kind, but the *projective continuous groups* may also be constructed, as well as the *compact spectral homology groups.*

Spectral and projective groups of nerves of *arbitrary* coverings form direct and inverse systems of compact or discrete groups, whose limit groups (in the sense of § 1) are *Čech groups* of a space; in particular, we obtain *spectral and projective homology groups with compact coefficients.*

Groups of vietorisian complexes of coverings[2]) with homomorphisms, induced by inclusion maps, form inverse and direct systems, whose limit groups are spectral and projective *Vietoris groups* of a space over discrete or *compact* coefficients.

The relations between the spectral and projective groups, stated above for complexes, extend to the groups of spaces in any homology theory – singular, Vietoris, etc. The relations between various theories, established previously for cases when projective and spectral groups coincide (for Čech and Vietoris theories in [7b], for singular and continuous theories in [9], for singular and Čech theories in [13], for Čech and Alexander-Kolmogoroff theories in [6a, e] and [12]), are valid for other

[1]) Proved by P. S. Alexandroff [2a] for sequences and generalised by the present author [6a] for arbitrary systems; cf. [4, 12].

[2]) By the vietorisian of a covering we mean a complex, whose vertices are points of the space, a subset of vertices forming a simplex, if and only if it is contained in an element of the covering.

cases likewise, provided the groups in question are of the same kind – projective or spectral.

Of the various applications which these groups have already received in topology and variational calculus [1; 2b, c; 3; 6b–g; 14; 15], we shall consider here the duality laws, and not only in view of their classical character, but in view of also the distinguished rôle, EDUARD ČECH's investigations play in this field.

3. Duality theorems for spectral groups. Let S^n be an n-sphere, A an arbitrary set of S^n, F_a a compact subset of A and G_a the complement of F_a. Let, further, K be a triangulation of G_a, K_α a finite subcomplex of K, and L_α the triangulated complement in S^n of K_α. The Alexander-Pontrjagin theorem asserts the duality

$$H_r(L_\alpha, X) \mid H_{n-r-1}(K_\alpha, Y), \tag{3}$$

where $H_s(M, Z)$ denotes the s-dimensional homology group of M over a group of coefficients Z, and X and Y are dual, $X \mid Y$. In its original form it was necessary to suppose in this theorem X to be a *discrete* group. But interpreting $H_r(L_\alpha, X)$ as a spectral group, we can extend this theorem to the case when X is *compact*. On carrying out this extension we are faced, for the first time, with the necessity of applying the homological approximations of a set by its compact subsets and, *simultaneously*, the approximations of its complement by the complements of the compact subsets just mentioned. The approximation of L_α by its finite closed subcomplexes $L_{\alpha\tau}$ gives precisely the spectral group $H_r(L_\alpha, X)$ which is the limit of the system

$$\{H_r(L_{\alpha\tau}, X), i_{\tau\sigma *}\}, \tag{4}$$

where $i_{\tau\sigma *}$ are the homomorphisms induced by the inclusions $i_{\tau\sigma} : L_{\alpha\tau} \to L_{\alpha\sigma}$, $\tau < \sigma$.

The homology groups $H_{n-r-1}(K_{\alpha\tau}, Y)$ of the complements

$$K_{\alpha\tau} = S^n \setminus L_{\alpha\tau}$$

with the homomorphisms $j_{\sigma\tau *}$, induced by the inclusions $j_{\sigma\tau} : K_{\alpha\sigma} \to K_{\alpha\tau}$, form an inverse system of groups

$$\{H_{n-r-1}(K_{\alpha\tau}, Y), j_{\sigma\tau *}\}. \tag{5}$$

Since, by Alexander-Pontrjagin duality theorem in its original form, i. e. when Y is discrete, the groups

$$H_r(L_{\alpha\tau}, X) \quad \text{and} \quad H_{n-r-1}(K_{\alpha\tau}, Y)$$

are dual, systems (4) and (5) are dually paired. Hence the limit-groups of (4) and (5), if the limits are taken in the sense of § 1, are dual. But the limit-group of (5) is isomorphic to the limit-group of the inverse system

$$\{H_{n-r-1}(N_\xi, Y), \omega_{\eta\xi}\}, \tag{6}$$

where N_ξ are nerves (or vietorisian complexes) of external open coverings U_ξ of K_α (i. e. U_ξ is a system of open sets of S^n, whose union contains K_α) and $\omega_{\eta\xi}$ are the corresponding homomorphisms of the homology groups. But the external coverings can be substituted in (6) by the internal coverings of K_α (i. e. by the coverings of K_α by *its* open subsets), in virtue of the following lemma of Čech (for proofs of different forms of this lemma under various conditions cf. [5; 11; 14; 15; 6b]):

Čech's lemma. *For any external covering U and any internal covering u of A there exist isomorphic* (i. e. with isomorphic nerves) *external and internal coverings V and, respectively, v of A, which are refinements of U and u, respectively, and satisfy the condition* $v = V \cap A$.

Applying this lemma, *in the case when A is a polyhedron*, to system (6) we conclude at once that the limit-group of (6) is isomorphic to the homology group $H_{n-r-1}(K_\alpha, Y)$ of K_α. Thus, the *Alexander-Pontrjagin duality* (3) *holds also when X is a compact group.*

Let us consider now the cohomology groups $H^r(L_\alpha, Y)$ and $H^{n-r-1}(K_\alpha, X)$; understanding $H^r(L_\alpha, Y)$ as the spectral group of L_α, we see that these cohomology groups are character groups of the corresponding homology groups under discussion, and we obtain the diagram:

$$(7) \qquad \begin{array}{ccc} H_r(L_\alpha, X) & \nwarrow \mid \nearrow & H_{n-r-1}(K_\alpha, Y) \\ \hline H^r(L_\alpha, Y) & \swarrow \mid \searrow & H^{n-r-1}(K_\alpha, X) \end{array}.$$

A *diagram* of this kind has the following sense. It is a quadruple of graded groups which are connected by group multiplications and homomorphisms, denoted by $|$ and $\rightarrow$ respectively, and which are represented as the vertices of a square. The components of each pair of these graded groups are in a certain $1 - 1$-correspondence, which will be called the *correspondence of the diagram*. We shall consider not only the usual correspondence, when the *difference* of dimensions of the corresponding components is constant (*degree* of correspondence), but also a correspondence, such that the *sum* of dimensions of corresponding components is constant; this constant we call the *σ-degree.*

The corresponding components of any pair of neighbouring groups are paired to κ. We shall consider here the case, when one of the neighbouring groups is discrete, the other compact, and the multiplication is distributive, continuous and orthogonal. The corresponding components of any pair of opposite (non-neighbouring) groups are isomorphic. The multiplications and isomorphisms are *compatible* in the sense that: (a) the composition of any two correspondences of the diagram is a correspondence of the diagram; (b) x and y being corresponding elements of an arbitrary pair of isomorphic groups, and t an element of either of the two other groups of the diagram, $(t, x) = (t, y)$. Diagram (7) has all these properties; its correspondences are of degree 0 and of σ-degree $n - 1$.

By a *directed system of diagrams* $\{\mathscr{D}_\alpha\}$ we mean a set of diagrams $\mathscr{D}_\alpha$ indexed by a directed system $\{\alpha\}$ and satisfying the following conditions:

(a) for each α, β the quadruples of $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta$ are bijective;

(b) for each $\alpha < \beta$ the groups corresponding to each other by the bijection just mentioned are connected by homomorphisms in such a manner, that they form an inverse or direct system of groups; in the sequel these systems will be called systems *generated* by $\{\mathscr{D}_\alpha\}$;

(c) neighbouring systems of groups generated by $\{\mathscr{D}_\alpha\}$ have opposite directions and are paired to κ;

(d) the homomorphisms of opposite systems of groups, generated by $\{\mathscr{D}_\alpha\}$, commute with the isomorphisms of the diagrams $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta$.

The *limit diagram* $\mathscr{D}$ of the system $\{\mathscr{D}_\alpha\}$,

$$\mathscr{D} = \lim \{\mathscr{D}_\alpha\},$$

is a quadruple, consisting of the limit groups of the systems generated by $\{\mathscr{D}_\alpha\}$. These limit groups are in the same categories as the groups of corresponding systems, the definition of limit being as in § 1. The correspondanoes, multiplications, and isomorphisms of $\mathscr{D}$ are defined from those of $\mathscr{D}_\alpha$. For instance, if $p = \{p_\alpha\}$ and $q = \{q_\alpha\}$ are elements of limit-groups, then the multiplication (p, q) is defined as (p_α, q_α), the latter being independent of the choice of α. Now, the compatibility and other properties of diagrams mentioned above can be proved to be valid for $\mathscr{D}$. Thus, lim $\{\mathscr{D}_\alpha\}$ is a diagram. Degrees of correspondences of $\mathscr{D}$ coincide with those of $\mathscr{D}_\alpha$.

Diagram (7) is a diagram $\mathscr{D}_\alpha$ in the sense just described. If $\alpha < \beta$, i. e. $K_\alpha \subset K_\beta$, the groups of $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta$ are connected by the homomorphisms $i_{\alpha\beta *}, i^*_{\beta\alpha}, j_{\alpha\beta *}, j^*_{\beta\alpha}$, induced by the inclusions

$$i_{\alpha\beta} : K_\alpha \to K_\beta$$

and

$$j_{\beta\alpha} : L_\beta \to L_\alpha .$$

It can be verified that the set of all $\mathscr{D}_\alpha$ with homomorphisms $i_{\alpha\beta *}$, etc., is a directed system of diagrams $\{\mathscr{D}_\alpha\}$. This system homologically approximates to the set $G_a = K$ by its finite closed subcomplexes and to the set F_a by its neighbourhoods.

The limit diagram $\mathscr{D}_a = \{\mathscr{D}_\alpha\}$ consists of the spectral homology and cohomology groups of G_a, $H_{n-r-1}(G_a, Y)$ and $H^{n-r-1}(G_a, X)$ respectively, and of the external groups $H_r(F_a, X)$ and $H^r(F_a, Y)$ of F_a. But applying the lemma of Čech, *in the case when A is a compact set F_a*, we conclude as above, that the latter groups may be considered as usual, i. e. internal, Čech (or Vietoris) groups of F_a. Thus we obtain the diagram $\mathscr{D}_a$:

$$(8) \qquad \frac{H_r(F_a, X)}{H^r(F_a, Y)} \; \overset{1}{\underset{1}{\times}} \; \frac{H_{n-r-1}(G_a, Y)}{H^{n-r-1}(G_a, X)},$$

representing the *correlations between the spectral groups of a compact—open pair of complementary sets* (F_a, G_a).

Now let us consider such diagrams for each a, i. e. for each compact subset F_a of A, and let us connect them for each $a < b$, i. e. for each $F_a \subset F_b$, by homomorphisms induced by the inclusions

$$i_{ab} : F_a \to F_b \quad \text{and} \quad j_{ba} : G_b \to G_a .$$

The set of all diagrams $\mathcal{D}_a$ and homomorphisms just mentioned form a directed system of diagrams, giving the internal homological approximation of A by its compact subsets F_a and, simultaneously, the external approximation of B by complements G_a of F_a. The limit diagram

$$\mathcal{D} = \lim \{\mathcal{D}_a\}$$

consists of the homology and cohomology groups of A with compact carriers, $H_r(A, X)$ and $H^r(A, Y)$ respectively, and of the external groups $H_{n-r-1}(B, Y)$ and $H^{n-r-1}(B, X)$ of B. But the lemma of Čech for *arbitrary* A guarantees that the latter groups can be understood not only as limit-groups of the system consisting of groups of neighbourhoods or of groups based on external coverings, but as usual Čech (or Vietoris) groups of B. Here it must be taken into account that these groups must be based not on *finite* coverings of B, as originally defined for the Čech groups [5], but on *all* open coverings of B [2b; 6c, d; 7a; 11]. These groups may be considered also as Vietoris groups [7b] and, when A is a neighbourhood retract (in particular, when A is an infinite polyhedron [6b]) as singular groups (see [13], cf. [6b]). Thus we obtain the diagram $\mathcal{D}$

$$\frac{H_r(A, X)}{H^r(A, Y)} \times \frac{H_{n-r-1}(B, Y)}{H^{n-r-1}(B, X)}, \tag{9}$$

which gives the various forms of the *Alexander-Pontrjagin duality theorem for an arbitrary pair of sets* (A, B). Certain of these forms and their particular cases (theorems for external groups, for the discrete group of coefficients X, for the compact group of coefficients X, etc.) have been obtained by P. S. Alexandroff, N. A. Berikashvili, A. N. Kolmogoroff, K. A. Sitnikov and G. S. Chogoshvili [2b, c; 3; 6b, c, d, f; 14]. The diagram $\mathcal{D}$, especially the compatibility of $\mathcal{D}$, and the way it was obtained above, show the interrelations of these forms with each other, and prove that all of them can be obtained by one and the same method, the chief tools being: *the simultaneous approximations to the sets by compact subsets and their complements, the theory of group systems, and Čech's lemma.*

The duality of the first line of $\mathcal{D}$ — the earliest to have been obtained — gives the Alexander-Pontrjagin theorem in its classical form. Moreover, it is the form from which it is easiest to obtain the duality theorems for non-closed sets obtained previously, namely Eilenberg's theorems relating to cases: (a) when $r = 0$ and n is arbitrary, and (b) when $n = 2$ and A is a homeomorphic image of a linear set (cf. [2b; 6b]).

The proof of the isomorphism of external and internal groups, given by P. S. Alexandroff [2], differs from that sketched above. Alexandroff's proof makes use of the canonical triangulations and transformations, which proved to be very useful in the generalisation of duality theorems for projective groups.

4. Duality theorems for projective groups. The relations which, in this case, constitute the starting point are represented by the following diagram $\mathcal{D}_a$:

$$\frac{H_r(F_a, X)}{H^r(F_a, Y)} \times \frac{H_{n-r}(G_a, Y)}{H^{n-r}(G_a, X)}. \tag{10}$$

Here $H_r(F_a, X)$ and $H^r(F_a, Y)$ are r-dimensional Steenrod's homology and, respectively, cohomology groups of regular cycles and cocycles of F_a, while $H_{n-r}(G_a, Y)$ and $H^{n-r}(G_a, X)$ are projective homology and cohomology groups of $G_a = K$. The isomorphisms and horizontal dualities of (10) are forms of Steenrod's duality theorem [16]. From these forms, Steenrod's original form can be obtained by applying Poincaré's duality theorem to the groups of K. The groups which we obtain by this dualisation form, with the groups of left and right verticals of $\mathscr{D}_a$, two auxiliary diagrams. The vertical dualities of $\mathscr{D}_a$ are ordinary dualities of the homology and cohomology groups. The duality of the right vertical was considered in § 2. The duality of the left vertical is obtained similarly: in each complex participating in the definition of Steenrod's groups we only need interchange chains and cochains, i. e. consider the inverse system of chains and the direct system of cochains of finite open subcomplexes in order to form the corresponding projective groups of the complex; when Y is compact, the limit is taken in the sense of § 1.

The dualities and isomorphisms mentioned above satisfy the conditions of § 3 and, therefore, $\mathscr{D}_a$ is a diagram. Its correspondences are of degree 0 and of σ-degree n.

Consider now the set of all diagrams $\mathscr{D}_a$, ordered by

$$a < b \Leftrightarrow F_a \subset F_b ,$$

and the set of all homomorphisms of the groups of $\mathscr{D}_a$ and $\mathscr{D}_b$, $a < b$, induced by the inclusions

$$i_{ab} : F_a \to F_b \quad \text{and} \quad j_{ba} : G_b \to G_a .$$

These diagrams and homomorphisms form a directed system of diagrams $\{\mathscr{D}_a\}$. To prove this, it is most convinient to use the auxiliary diagrams mentioned above. Supplementary groups of the auxiliary diagrams are connected by homomorphisms induced by canonical transformations (see end of § 3).

The limit diagram of the system $\{\mathscr{D}_a\}$ is

$$(11) \qquad \frac{H_r(A, X)}{H^r(A, Y)} \begin{smallmatrix} \mid \\ \nwarrow\nearrow \\ \swarrow\searrow \\ \mid \end{smallmatrix} \frac{H_{n-r}(B, Y)}{H^{n-r}(B, X)} .$$

Here $H_r(A, X)$ and $H^r(A, Y)$ are Steenrod's homology and cohomology groups of A with compact carriers. The groups $H_{n-r}(B, Y)$ and $H^{n-r}(B, X)$ are the homology and, respectively, the cohomology groups of B, based on neighbourhoods of B. But, as above, it can easily be shown that these groups coincide with the groups based on external coverings and, consequently, in virtue of Čech's lemma, with projective Čech groups of B. (It is to be noted that there does not exist an invariant definition of the limit groups of systems consisting of supplementary groups of the auxiliary diagrams). Thus, diagram (11) gives the duality theorems for projective groups of arbitrary pairs of sets (A, B). The isomorphism of the groups $H_r(A, X)$ and $H^{n-r}(B, X)$ is the generalisation of Steenrod's duality theorem which coincides with the theorem proved by K. A. Sitnikov, Steenrod's groups being isomorphic to the groups considered by K. A. Sitnikov (see [14], cf. [3; 6g]).

If X is compact, we obtain, from the coincidence of the spectral and projective groups, the coincidence of diagrams (9) and (11) and, therefore, of Steenrod's and Vietoris' groups; in this case the two coinciding theorems constitute the theorem of Alexander-Pontrjagin in its original form.

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