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CONNEXIONS BETWEEN CONVEXITY OF A METRIC CONTINUUM X AND CONVEXITY OF ITS HYPERSPACES $C(X)$ AND 2^X

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Let X be a continuum with a metric ϱ. We denote by $C(X)$ the hyperspace of all nonvacuous subcontinua of X and by 2^X the hyperspace of all nonvacuous and closed subsets of X, both metrized by the Hausdorff metric ([2], p. 291):

$$\varrho^1(A, B) = \max\left[\sup_{a\in A} \varrho(a, B), \sup_{b\in B} \varrho(A, b)\right]. \tag{1}$$

By virtue of a theorem of Mazurkiewicz ([5], see also [3], th. 2.7) the hyperspaces $C(X)$ and 2^X are continua.

For every subset Z of X and every $\eta \geqq 0$ let $Q(Z, \eta)$ be a generalized solid sphere of centre Z and radius η, i. e.

$$Q(Z, \eta) = \{x : x \in X, \varrho(x, Z) \leqq \eta\}.$$

The formula

$$\varrho^1(A, B) = \inf\{\eta : [A \subset Q(B, \eta)], [B \subset Q(A, \eta)]\} \tag{2}$$

is equivalent to the formula (1) ([3], p. 22).

We use in this paper the notion of convexity in the well known general sense of K. Menger ([6], p. 81): A space X is said to be convex provided that for each two distinct points x, y of X there exists a point $z \in X$ different from x and y which lies between x and y, i. e.

$$\varrho(x, y) = \varrho(x, z) + \varrho(z, y).$$

It is known ([6], p. 89, see also [1]) that in complete convex spaces X each pair of points $x, y \in X$ is joined by a metric segment $\overline{xy}$, i. e. by a subset of X isometric to a segment of the real line with length $\varrho(x, y)$. A space X will be said to be strongly convex provided that each two points $x, y \in X$ are joined by precisely one metric segment.

Let A and B be two subsets of X such that for each pair of points $a \in A$ and $b \in B$ there is at least one segment $\overline{ab}$ in X. We shall call a *junction in X between A and B*, denoting it by $J(A, B)$, the union of these segments, i. e. the set containing with each pair of points $a \in A$ and $b \in B$, at least one segment $\overline{ab}$. We shall call a *bridge in X between A and B*, denoting it by $P(A, B)$, every compact junction in X between A and B.

We shall use the following four lemmas:

Lemma 1. *If X is a metric continuum, A and B closed subsets of X, and $J(A, B)$ a junction in X between A and B, then its closure $\overline{J(A, B)}$ is the bridge in X between A and B.*

Lemma 2. *If X is a metric space, A and B closed subsets of X, $P(A, B)$ a bridge in X between A and B, and if $H \subset P(A, B)$ is closed, then there exist bridges $P(A, H)$ and $P(H, B)$, both contained in $P(A, B)$.*

Lemma 3. *If X is a metric space, A and B closed subsets of X such that there exists a bridge $P(A, B)$ in X between A and B, and if ε is a number such that $0 \leqq \varepsilon \leqq \varrho^1(A, B)$, then the set*

$$H = P(A, B) \cap Q(A, \varepsilon) \cap Q[B, \varrho^1(A, B) - \varepsilon] \tag{3}$$

satisfies the conditions $H \in 2^X$, $\varrho^1(A, H) = \varepsilon$, $\varrho^1(H, B) = \varrho^1(A, B) - \varepsilon$.

Lemma 4. *If X is a convex metric continuum and if every subcontinuum of X is convex, then the following sets are strongly convex: the continuum X, the generalized solid sphere $Q(A, \varepsilon)$ for every continuum $A \subset X$ and every $\varepsilon \geqq 0$, the bridge $P(A, B)$ for every pair of subcontinua A and B of X.*

We have the following six theorems, the proofs of which will be outlined only:

Theorem 1. *If X is a metric continuum, A and B closed subsets of X and if there is a bridge $P(A, B)$, then there exists in 2^X at least one segment between A and B. If, moreover, A and B are subcontinua of X and every subcontinuum of X is convex, then there exists in $C(X)$ at least one segment between A and B.*

In fact, let $A = H_0 \in 2^X$ and $B = H_1 \in 2^X$. By Lemma 3 for $\varepsilon = 2^{-1} . \varrho^1(H_0, H_1)$ there exists a set $H_{1/2}$ defined by formula (3), i. e. such that $H_{1/2} \subset P(H_0, H_1)$ and

$$\varrho^1(H_0, H_{1/2}) = \varrho^1(H_{1/2}, H_1) = 2^{-1} . \varrho^1(H_0, H_1) .$$

By induction and using lemmas 2 and 3, we define a family of closed sets $\{H_{k/2^n}\}$, where $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$, with the following properties:

$$H_{2k/2^{n+1}} = H_{k/2^n} \quad \text{for} \quad n = 0, 1, \ldots \quad \text{and} \quad k = 0, 1, \ldots, 2^n ,$$
$$H_{(2k+1)/2^n} \subset P(H_{2k/2^n}, H_{(2k+2)/2^n}) \quad \text{for} \quad k = 0, 1, \ldots, 2^{n-1} - 1 ,$$
$$\varrho^1(H_{k/2^n}, H_{m/2^n}) = \frac{|k - m|}{2^n} \varrho^1(H_0, H_1) \quad \text{for} \quad k, m = 0, 1, \ldots, 2^n .$$

The closure in 2^X of this family is a segment between A and B ([6], p. 87–89).

If, moreover, every subcontinuum of X is convex, $A = H_0 \in C(X)$ and $B = H_1 \in C(X)$, then the set $H_{1/2}$ is a continuum (strongly convex) since, by (3), it is the intersection of three continua which are strongly convex by Lemma 4 ([6], p. 104). Hence $H_{1/2} \in C(X)$. For the same reason each $H_{k/2^n}$, where $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$, is a continuum. Therefore the closure in $C(X)$ of the family $\{H_{k/2^n}\}$ is a segment in $C(X)$ between A and B ([6], p. 87–89 and [4], p. 110).

The Theorem 1 implies at once the following

Theorem 2. *If X is a convex metric continuum and every subcontinuum of X is convex, then the hyperspace $C(X)$ is convex.*

The converse implication is an open problem (see p. 145).

Theorem 3. *If X is a metric continuum and at least one of the hyperspaces $C(X)$ and 2^X is convex, then X is convex.*

In fact, let $p \in X$ and $q \in X$. At least one of the hyperspaces $C(X)$ and 2^X being convex by hypothesis, there exists in this hyperspace a segment between (p) and (q), composed of subsets of X. Therefore the inequality $0 \leqq \varepsilon \leqq \varrho^1(p, q)$ implies the existence of a set $Z \subset X$ belonging to this segment and such that $\varrho^1(p, Z) = \varepsilon$ and $\varrho^1(Z, q) = \varrho^1(p, q) - \varepsilon$.

As can be seen easily, each point $z \in Z$ satisfies the equalities $\varrho(p, z) = \varepsilon$ and $\varrho(q, z) = \varrho(p, q) - \varepsilon$. Hence the continuum X is convex.

Theorem 4. *If X is a metric continuum containing isometrically the boundary of a square, then the hyperspace $C(X)$ is not convex.*

In fact, let $K \subset X$ be a continuum isometric with the boundary of the unit square in the plane $0xy$, with opposite vertices $(0, 0)$ and $(1, 1)$. The continuum K is then a union of 4 segments: I with ends $(0, 0)$ and $(0, 1)$, II with ends $(0, 1)$ and $(1, 1)$, III with ends $(1, 1)$ and $(1, 0)$, and IV with ends $(1, 0)$ and $(0, 0)$. Consider the continua $A = \mathrm{I} \cup \mathrm{II} \cup \mathrm{IV}$ and $B = \mathrm{III} \cup \mathrm{II} \cup \mathrm{IV}$. We have

$$Q(A, 4^{-1}) \cap Q(B, 4^{-1}) = Q(\mathrm{II}, 4^{-1}) \cup Q(\mathrm{IV}, 4^{-1}), \tag{4}$$

$$Q(\mathrm{II}, 4^{-1}) \cap Q(\mathrm{IV}, 4^{-1}) = 0. \tag{5}$$

Since $\varrho^1(A, B) = 2^{-1}$, it suffices to prove that there exist no continuum $H \subset X$ such that

$$\varrho^1(A, H) = \varrho^1(H, B) = 4^{-1}. \tag{6}$$

Note first the following implication: each of the inclusions

$$H \subset Q(\mathrm{II}, 4^{-1}) \quad \text{and} \quad H \subset Q(\mathrm{IV}, 4^{-1}) \tag{7}$$

implies both of the inequalities

$$\varrho^1(A, H) > 4^{-1} \quad \text{and} \quad \varrho^1(H, B) > 4^{-1}. \tag{8}$$

Suppose now that there exists a continuum $H \subset X$ satisfying (6). Then by (2) we have $H \subset Q(A, 4^{-1})$ and $H \subset Q(B, 4^{-1})$, whence $H \subset Q(A, 4^{-1}) \cap Q(B, 4^{-1})$. By (4) and (5) there follows one of the inclusions (7), and therefore, by the mentioned implication, the inequalities (8), contrary to (6).

Theorem 5. *If X is a metric continuum, then the hyperspace 2^X is convex if and only if X is convex.*

In fact, if the hyperspace 2^X is convex, then by Theorem 3 the continuum X is also convex. Inversely, if the continuum X is convex, then evidently there exists, by the definition of junction, a junction $J(A, B)$ in X between A and B for each two closed subsets A and B of X. Then by Lemma 1 there exists a bridge $P(A, B)$ in X between A and B, and by Theorem 1 there follows the existence of a segment in 2^X joining A and B. Hence 2^X is convex.

Theorem 6. *If a metric continuum X can be immersed isometrically in Euclidean n-space E^n with $n \geqq 1$, and if the hyperspace $C(X)$ is convex, then X is a segment — and conversely.*

In fact, if X is a segment, then every subcontinuum is a segment or a point and therefore is convex. Hence by Theorem 2 the hyperspace $C(X)$ is also convex.

Conversely, if $C(X)$ is convex, then X is convex by Theorem 3 and contains no boundary of a square by Theorem 4. Therefore dim $X \leqq 1$, because every convex and at least 2-dimensional continuum $X \subset E^n$ contains some square. The only 1-dimensional convex continuum lying isometrically in Euclidean space is a segment, of course.

Problems. 1. *Characterize the family of continua whose hyperspaces of subcontinua are convex.*

This problem was solved for continua isometrically immersible in Euclidean space only: the characteristic property is to be a segment.

Among continua which are not isometrically immersible in Euclidean spaces, the dendrites (i.e. acyclic and locally connected continua), metrized by arc-length have only convex subcontinua and therefore, by Theorem 2, their hyperspaces of subcontinua are convex.

The solution of problem 1 would be obtained from the positive answer to the following problem (see Theorem 2):

2. *Does the convexity of the hyperspace $C(X)$ of a continuum X imply that every subcontinuum of X is convex?*

Remark. Detailed proofs of lemmas and theorems formulated here are contained in the author's article "On convex metric spaces III." Fundamenta Mathematicae *51* (1962).

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