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DESCRIPTIONS OF ČECH COHOMOLOGY[1])

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E. H. SPANIER [5] proved that, for compact spaces, a form of the Alexander-Kolmogoroff homology theory suggested by A. D. WALLACE was isomorphic to the Čech theory. W. HUREWICZ, J. DUGUNDJI and C. H. DOWKER [4] established this result for paracompact spaces, and Dowker [3] later proved isomorphism for arbitrary topological spaces. P. ALEXANDROFF has an unpublished proof of the same theorem. The purpose of this note, largely methodological, is to outline in some detail a proof of isomorphism for paracompact spaces. It is remarkable that the proof is completely elementary and non-combinatorial in character. The corresponding development for homology with coefficients in a sheaf is sketched without proof in the last section.

Čech Cohomology. We review the definition of the Čech cohomology groups of a space X with coefficient group G in order to establish the notation. Suppose $U = \{U(i)\}_{i \in I}$ is an (indexed) open cover of X. For each $(q + 1)$-tuple $s = (s_0, s_1, \ldots, s_q)$ of members of the index set I, we let $|U(S)|$ be the intersection $\bigcap\{U(s_i) : i = 0, 1, \ldots, q\}$, and we define the nerve of the cover $U = \{U(i)\}_{i \in I}$ to be the complex with q-dimensional simplices $K_q(U) = \{s : |U(s)| \text{ non-void}\}$. The q-dimensional cochain group $C^q(U)$ is $\{f : f \text{ is a function on } K_q(U) \text{ to } G\}$, and the usual coboundary operator on $C^q(U)$ to $C^{q+1}(U)$ then defines the cohomology groups $H^q(U)$ of the cover.

If $V = \{V(j)\}_{j \in J}$ is also an open cover of X then we say that V is a refinement of U iff $V(j) \subset U(n_j)$ for some suitably chosen function n on J to I. We call n a refining function; n induces a refining map on $K_q(V)$ to $K_q(U)$, which in turn induces a chain map on $C^q(U)$ to $C^q(V)$, and this chain map induces a refining homomorphism of $H^q(U)$ into $H^q(V)$. This homomorphism is independent of the particular refining function n which is chosen. The Čech cohomology group $H^q(X)$ is defined to be the inductive limit (direct limit), under the refining homomorphisms, of the groups $H^q(U)$ for all open covers U of X.

Small Simplex Cohomology (Vietoris Type). There is a special sort of cover which is of particular interest to us. Suppose that N is an open subset of the product $X \times X$ which contains the diagonal $\Delta = \{(x, x) : x \in X\}$. For each member x of X we define $N[x]$ to be $\{y : (x, y) \in N\}$ and we denote by N^* the cover $\{N[x] : x \in X\}$. Thus the space X itself is the index set for the cover N^*. It is known that, in case X is paracom-

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pact, every open cover has a refinement which is of the form N^*. In other words, the class of covers of the form N^* is cofinal in the class of all open covers of X. The class of open neighborhoods of Δ is directed by $\subset$, and we notice that if M and N are open neighborhoods of the diagonal and $M \subset N$ then the cover M^* is a refinement of the cover N^*. Moreover, if $M \subset N$ then there is a natural choice for the refining function which carries the index set X of M^* into the index set X of N^*, namely the identity. The set $K_q(M^*)$ of q-simplices of the cover M^* is in fact a subset of $K_q(N^*)$, and the induced refining chain map of $C^q(N^*)$ into $C^q(M^*)$ is restriction; that is the image of $f \in C^q(N^*)$ is $f \mid K_q(M^*)$. It follows from these facts that the Čech group $H^q(X)$ is isomorphic to the inductive limit under the homomorphism induced by restriction of $H^q(N^*)$ for neighborhoods N of the diagonal in $X \times X$.

The preceding description of Čech cohomology has a natural geometric interpretation. If we agree that a simplex $(x_0, x_1, \ldots, x_q)$ with vertices in X is N-small if $\bigcap\{N[x_i] : i = 0, 1, \ldots, q\}$ is non-void, then $K_q(N^*)$ is just the set of N-small q-simplices, so that the cohomology theory may be called a "small simplex" theory.

Alexander-Kolmogoroff Cohomology. We next need the fact "cohomology commutes with inductive limit". More precisely: let $C^q(X)$ be the inductive limit, under the restriction maps, of $C^q(N^*)$ for N a neighborhood of the diagonal in $X \times X$. The coboundary operator on the cochain groups $C^q(N^*)$ induces a coboundary operator on $C^q(X)$, and thus defines a cohomology group which we may denote $^*H^q(X)$. It is not hard to see that $^*H^q(X)$ is isomorphic to the Čech group $H^q(X)$, since each is isomorphic to a group which can be described informally as $\{f : f \in C^q(N^*)$ for some N, and for some M the restriction of f to $K_q(M)$ is a cocycle$\}$ modulo the equivalence relation $\{(f, g) :$ for some neighborhood P of the diagonal, $f \mid K_q(P) - g \mid K_q(P)$ is a coboundary$\}$.

We are now very close to the Alexander-Kolmogoroff cohomology theory. The set $K_q(N^*)$ is the subset of the set $X^{(q+1)}$ consisting of all $(q + 1)$-tuples of points of X which are N-small. Thus $K_q(N^*)$ is a neighborhood of the diagonal $\Delta^{(q+1)} = \{(x_0, x_1, \ldots, x_q) : x_i = x_0$ for all $i\}$, and we shall refer to $K_q(N^*)$ as the N-neighborhood of $\Delta^{(q+1)}$. The inductive limit $C^q(X)$ of the groups $C^q(N^*)$ is then, by reason of the definition of the inductive limit, the set $\{(f, N) : f$ on the N-neighborhood of $\Delta^{(q+1)}$ to $G\}$, modulo the equivalence relation: (f, N) is equivalent to (g, M) iff for some $P, f = g$ on the P-neighborhood of $\Delta^{(q+1)}$. Because the space X is paracompact, the family of N-neighborhoods of $\Delta^{(q+1)}$ is a base for the family of all neighborhoods of $\Delta^{(q+1)}$, and consequently $C^q(X)$ is isomorphic to the family F^q of all functions f, each defined on some neighborhood of $\Delta^{(q+1)}$ to G, modulo the subset of all functions f which vanish on some neighborhood of $\Delta^{(q+1)}$. (The isomorphism carries each equivalence class belonging to $C^q(X)$ into the equivalence class containing it.) Finally, each equivalence class of F^q clearly contains members with domain equal to $X^{(q+1)}$. Whence: The Čech cohomology group $H^q(X)$ is isomorphic to the cohomology group of the chain complex with q-dimensional cochain group equal to the group of all functions on $X^{(q+1)}$ to G, modulo the subgroup consisting of functions zero on some neighborhood of the diagonal $\Delta^{(q+1)}$. This is the Alexander-Kolmogoroff cohomology theory.

Cohomology with Coefficients in a Sheaf. Essentially the same reasoning as that given above yields a description of Alexander-Kolmogoroff type for the Čech cohomology group of a paracompact space X with coefficients in a sheaf $\mathscr{I}$ of Abelian groups over X. Let Σ be the set of all sections of $\mathscr{I}$, where sections are added pointwise, the domain of the sum of two sections being the intersection of the domains. Let C^q be the set of all functions f on $X^{(q+1)}$ to Σ with the property that for each member x of X there is a neighborhood U of x such that if $s \in U^{(q+1)}$ then U is a subset of domain of $f(s)$. Let R^q be the equivalence relation: $R^q = \{(f, g) :$ for $x \in X$ there is a neighborhood U of x such that $f(s) \mid U = g(s) \mid U$ for $s \in U^{(q+1)}\}$. The quotient C^q/R^q inherits an addition from Σ, and with the natural coboundary operator, the q-th cohomology group of the chain complex with q-th cochain group C^q/R^q is isomorphic to the Čech group $H^q(X, \mathscr{I})$.

There are several variations of the above description which pretty evidently give the same cohomology groups. R. Deheuvels [2] has a related description of $H^q(X, \mathscr{I})$ in terms of objects which are "locally" functions on $X^{(q+1)}$.

Finally, the group C^q/R^q has a natural representation as a family of functions on X. We may describe this representation in terms of the construction above as follows. For each $x \in X$ define the equivalence relation R_q^x to be $\{(f, g)$: for some neighborhood U of x, if $s \in U^{(q+1)}$ then $f(s) \mid U = g(s) \mid U\}$. Clearly $R^q = \bigcap\{R_q^x : x \in X\}$, and the natural map F such that $F(f/R^q)(x) = f/R_x^q$ is therefore an isomorphism. The family of all functions $F(f/R^q)$ might well be called the group $\mathscr{A}^q$ of Alexander cochains on X. It evidently has the property: if a and b belong to $\mathscr{A}^q$ and $a(x) = b(x)$ then $a \mid U = b \mid U$ for some neighborhood U of x. It is true, but not obvious, that a function b which locally belongs to $\mathscr{A}^q$, in the sense that every point of X has a neighborhood in which b agrees with some member of $\mathscr{A}^q$, necessarily belongs to $\mathscr{A}^q$. In brief, $\mathscr{A}^q$ is a complete carapace in the sense of H. CARTAN [1].

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