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ABSTRACT DISTANCE AND NEIGHBORHOOD SPACES

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1. It is well known that one can topologize a given set E in very different ways. In fact, let $P(E)$ be the family of all subsets of E; then every one-valued map $\tau: P(E) \to P(E)$ defines a generalized topology on E. The set E with the generalized topology τ is an abstract space which we shall designate by (E, τ). Neighborhood spaces in the sense of M. FRÉCHET are characterized by the following three conditions: a) $\tau\Lambda = \Lambda$ (Λ is the void set); b) $\tau A \supset A$ for every $A \subset E$; c) if $A \supset B$ then $\tau A \supset \tau B$ for every $A, B \subset E$. If the map τ satisfies the well known axioms of C. KURATOWSKI (i.e. $(\tau_1)\ \tau\Lambda = \Lambda$; $(\tau_2)\ \tau A \subset A$; $(\tau_3)\ \tau(A \cup B) = \tau A \cup \tau B$; $(\tau_4)\ \tau\tau A = \tau A$), *topological* spaces are characterized by either of the following ways:

a) by open sets;
b) by closed sets;
c) by interiors of subsets in E;
d) by neighborhoods of the points in E;
e) by bases or subbases of the topology.

It should be remarked that every topological space is a neighborhood space in the sense of M. Fréchet, but the inverse is not true.

Let R^1_+ be the space of all real numbers $\geqq 0$, as a subspace of the space R^1 of all reals. One introduces a (pseudo)metric topology in E by means of a (pseudo)metric d, i. e. a map $d: E \times E \to R^1_+$ which satisfies the well known axioms of the (pseudo)-metric. A topological space is (pseudo)metrizable iff it is homeomorph of a (pseudo)-metric space. A series of necessary and sufficient conditions for a topological space to be metrizable are now known (P. S. ALEXANDROV, YU. M. SMIRNOV, J. NAGATA, R. H. BING, A. V. ARCHANGELSKIJ). Since A. H. STONE's important paper on para-compactness, the famous metrization problem of topological spaces was fully solved in the middle of this century. Furthermore, it is known that one can generalize some metric notions to a more general class of topological spaces, i. e. to the class of A. N. TYCHONOFF'S completely regular spaces. In fact, call any two classes of spaces topologically equivalent, iff each space of the first class is also a space of the second class, and vice versa. Then the class of completely regular spaces is topologically equivalent both to the class of δ-spaces (V. A. EFREMOVIČ, YU. M. SMIRNOV) and to the class of separated uniform spaces (A. WEIL). Let us recall that a δ-space on E is obtained by means of a binary relation on $P(E)$ verifying the axioms of Efremovič-Smirnov; a uniform space on E is obtained by means of a family of subsets of the cartesian product $E \times E$ which satisfies the axioms of uniform structures. Necessary and sufficient conditions for the metrizability of a uniform space (in slightly modified sense) are known (A. Weil) and there are solutions of the metrization problem for δ-spaces (YU. M. SMIRNOV, A. S. SHWARZ).

2. Due to the fact that there are topological spaces which are not metrizable, one can try to topologize a given set by means of more general "metrics", whose range is not the space R^1_+ of non-negative reals, but a non-void set M structured by structures less rich than that of the space of real numbers. Of course, in particular cases one obtains metric spaces. In a general sense, one can term such "metrics" as *abstract distances* ("écarts abstraits").

The author's method is the following: Let $f: E \times E \to M$ be an one-valued map of the cartesian product $E \times E$ of a given set E into a given non-void set M. For every $a \in E$, the element $f(a, a) \in M$ is defined; for every $a \in E$, let there be given a family Fa of subsets $X_{\lambda_a} \subset M$ containing $f(a, a)$, i. e. $f(a, a) \in X_{\lambda_a}$, where λ_a is any element of an index-set (λ_a) equipollent to the family Fa. We define the following neighborhood system on E:

$$W_{\lambda_a}(a) = \{b : b \in E \text{ and } f(a, b) \in X_{\lambda_a}\}\,, \quad \lambda_a \in (\lambda_a)\,, \quad a \in E\,. \tag{1}$$

By means of (1) as a neighborhood base on E, we thus obtain a neighborhood space in the sense of M. Fréchet and it is evident that very different neighborhood spaces can be defined in this manner. According to the terminology of M. FRÉCHET or G. KUREPA, we shall refer to the function f as an *abstract distance* or *M-distance* („M-écart").

Conversely, let $(E, \mathscr{V})$ be any given neighborhood space, where $\mathscr{V}$ is any one of its neighborhood bases.

1. Definition. *We shall say that the neighborhood space $(E, \mathscr{V})$ admits an abstract distance or an M-distance f, iff there is a set M and a function f such that the corresponding neighborhood system* (1), *in the sense of the above definitions, is equivalent to the neighborhood base $\mathscr{V}$.*

1. Theorem. *The class of neighborhood spaces is topologically equivalent to the class of abstract spaces defined by means of neighborhood systems* (1).

It suffices to show that every neighborhood space $(E, \mathscr{V})$ admits at least one abstract distance in the sense of definition 1. In fact, we can prove ([6], p. 109) that every neighborhood space admits at least two abstract distances, one antisymmetric, $f(x, y) = y$, $x, y \in E$, and the other symmetric, $f(x, y) = \{x, y\}$, where $\{x, y\}$ is the *set* whose elements are the coordinates x and y of the *ordered pair* $(x, y) \in E \times E$ as elements.

As an example, let E be any non-void set and, for every $a \in E$, let $Fa = \{X_\varepsilon : \varepsilon > 0\}$ = the neighborhood base of the origin 0 in the space R^1_+, the *abstract distance* f being defined as an one-valued map $f: E \times E \to R^1_+$ satisfying the requirement

$f(a, a) = 0$ for every $a \in E$. In this case we can call the function f a *real distance*, but not a (pseudo)metric in the usual sense. It is not difficult to see that the neighborhood space (E, τ), defined by the corresponding neighborhood base (1), satisfies the first axiom of countability and the axiom (τ_3). If the function f satisfies the axioms of (pseudo)metrics, the space (E, τ), thus obtained is precisely a (pseudo)metric space, and f is then a (pseudo)metric. On the other hand, in the same example, let E be R^1 with its usual topology; this is a metric space with the metric $d(x, y) = |x - y|$, $x, y \in R^1$. Since this is a neighborhood space, by Theorem 1 and its proof, R^1 admits the antisymmetric abstract distance $f(x, y) = y$, and the symmetric one $f(x, y) = \{x, y\}$. According to the definitions given above, one can see that the *antisymmetric abstract distance* of two *real numbers* x and y is a real number, positive, zero or negative.

It is possible to give some conditions for a space, defined by an abstract distance, to be a topological space and, in particular, a T_1-space ([6], p. 111). Nevertheless, we mention here only the next theorem which is concerned with uniform spaces.

2. Theorem. *Let $f : E \times E \to M$ be an one-valued map of the cartesian product $E \times E$ into M and let $t_0 \in M$ be a fixed point. For every $a \in E$, let $Fa = F$, where F is a given family of subsets $X_\lambda \subset M$, $\lambda \in (\lambda)$, each containing t_0. Then the class of uniform spaces is topologically equivalent to the class of spaces admitting an abstract distance f such that the following conditions are satisfied*:

1° $f(a, a) = t_0, \ a \in E$.

2° $f(b, a) = f(a, b), \ (a, b) \in E \times E$.

3° *For every $\lambda, \mu \in (\lambda)$ there exists an $\nu \in (\lambda)$ with the property $X_\nu \subset X_\lambda \cap X_\mu$.*

4° *For every $\lambda \in (\lambda)$ there exists an $\mu \in (\lambda)$ so that $f(a, b) \in X_\mu$ and $f(b, c) \in X_\mu$ implies $f(a, c) \in X_\lambda$, $a, b, c \in E$.*

The class of separated uniform spaces is topologically equivalent to the class of spaces admitting an abstract distance verifying the conditions 1°—4° as well as the requirement:

5° $f(x, a) \in \bigcap_{\lambda \in (\lambda)} X_\lambda \Rightarrow x = a, \ x, a \in E$.

The proof of Theorem 2 is given in [6], pp. 112—114. We mention only the definition of abstract distance f in proving that every uniform space $(E, \mathcal{U})$ admits an abstract distance in the sense of the definitions given above. Let $\mathcal{B}$ be the base of the uniform structure $\mathcal{U}$ of the uniform space $(E, \mathcal{U})$, and let Δ be the set $\{(a, a) : a \in E\} \subset E \times E$. Then the abstract distance f is defined as follows:

$$f(a, b) = \begin{cases} \Delta, & \text{if } (a, b) \in \bigcap_{V \in \mathcal{B}} V, \\ \{(a, b), (b, a)\}, & \text{if } (a, b) \notin \bigcap_{V \in \mathcal{B}} V, \end{cases} \tag{2}$$

where $\{(a, b), (b, a)\}$ is the *set* whose elements are the two ordered pairs (a, b) and (b, a). It should be noted that the second part of the Theorem 2 (that is, the part which is concerned with the separated uniform spaces) was first proved by G. Kurepa in [1].

In fact, the definition (2) of the abstract distance f is a slight modification of G. Kurepa's definition, in order to prove that *every uniform space* admits an abstract distance in the sense of our definitions.

Of course, it is possible to choose abstract distances other than those given above, according either to the nature of the problem considered or to taste. For example, if the range of the function $f: E \times E \to M$ is a subset of an ordered set (a scale) M, then one can obtain more interesting abstract distances which could be termed *ordered abstract distances* or simply *ordered distances*. Totally ordered distances were first introduced by G. Kurepa (1934, cf. [1]) in defining "espaces pseudo-distanciés". After the Second World War, M. Fréchet raised analogous problems; their investigation was continued by A. APPERT [2], and J. COLMEZ [3], thus leading to a characterization of separated uniform spaces by means of J. Colmez's abstract ordered distance of the first kind (cf. [3]). All mentioned ordered distances are special cases of the definitions given above.

It has come to my notice at this Symposium on General Topology in Prague that recently an abstract distance of the most importance has been introduced by the Russian mathematicians M. ANTONOVSKI, V. BOLTJANSKI and T. SARIMSAKOV in their interesting book [4]. Their results was reported by M. Antonovski. The authors first defined a topological semifield as an associative topological ring E containing a subset K verifying six axioms named *axioms of topological semifields* (cf. [4], § 1). The elements of K are termed the positive elements of the semifield E. By the authors' definition ([4], § 3), the set X is a *metric space* over the semifield E if there is given an one-valued map

$$\varrho : X \times X \to \overline{K} \tag{3}$$

which satisfies the following conditions:

1° $\varrho(x, y) = 0$ iff $x = y$.

2° $\varrho(x, y) = \varrho(y, x)$.

3° $\varrho(x, y) + \varrho(y, z) \gg \varrho(x, z)$, $x, y, z \in X$ ($a \gg b$ means that $a - b \in \overline{K}$).

A metric space X over the semifield E, with the metric ϱ, is denoted by (X, ϱ, E). In the special case of $E = R^1$ one obtains the usual metric space.

From the authors' theorem 11.1 (§ 3) in [4], it is not difficult to observe that the metric (3) is precisely a kind of an abstract distance in the sense of our definitions leading to (1). In fact, if U is any neighborhood of the zero point 0 in the topological semifield E, the topology in X is introduced by means of the neighborhoods

$$\Omega(x, U) = \{y : y \in X \text{ and } \varrho(x, y) \in U\}, \quad x \in X, \tag{4}$$

which are defined in the same manner as that of the definition of neighborhood systems (1). But the metric (3) is remarkable in that it is closely related to the usual real metric. This can be seen, for example, from theorem 11.3 (§ 3) in [4] by which the topology, introduced by means of (4), of the metric space (E, ϱ, E), where ϱ is the function $\varrho(x, y) = |x - y|$, $x, y \in E$, is identical with the topology of the topological semifield E.

On the other hand, by the authors' definition and theorem 11.5 (§ 3) in [4], the class of topological spaces metrizable over any topological semifield is topologically equivalent to the class of completely regular spaces. This last class being topologically equivalent either to the class of separated uniform spaces or to the class of δ-spaces, one can see that the metric ϱ, defined by (3), is a new abstract distance, more suitable than those mentioned above, characterizing the class of separated uniform spaces. In this way one can say that the class of completely regular spaces is indeed very close to the class of metrizable topological spaces (in the usual sense).

3. Let (E, τ) and (M, σ) be two abstract spaces. By G. Kurepa's definition (cf. [1]), the space (E, τ) is of class $\mathscr{E}[M]$, iff there is an one-valued map $f : E \times E \to M$ which satisfies the following conditions:

0^1 If $f(b, a) = f(a, a)$ then $b = a$, for every $a, b \in E$.

0^2 $f(b, a) = f(a, b)$, for every $a, b \in E$.

0^3 For $a \in E$ and $A \subset E$, let $f(a, A) = \{f(a, b) : b \in A\}$; then

$$\tau A = \{a : a \in E \text{ and } f(a, a) \in \sigma f(a, A)\},$$

where τA means $\bar{A}$ in the space (E, τ) and $\sigma f(a, A)$ means $\overline{f(a, A)}$ in the space (M, σ). One can say that f is an abstract distance verifying G. Kurepa's conditions 0^1, 0^2 and 0^3. We have the following theorem (cf. [6], p. 115):

3. Theorem. *Let $(M, \mathscr{V})$ be a neighborhood space, $\mathscr{V}$ being one of its neighborhood bases, and let $Fa = \mathscr{V} f(a, a)$, $a \in E$, where $\mathscr{V} f(a, a)$ is the local neighborhood base of the point $f(a, a)$ in the space $(M, \mathscr{V})$, in the definition (1). Then the abstract space (E, τ) is of class $\mathscr{E}[M]$, iff it is a neighborhood space and admits at least one M-distance f which satisfies G. Kurepa's conditions* 0^1 *and* 0^2.

Among spaces of the class $\mathscr{E}[M]$ we mention here only the class of R-spaces, defined by G. Kurepa [1] and P. Papić [5]. The space (E, τ) is an R-space iff the following conditions are satisfied:

1° The space (E, τ) has a neighborhood base $\mathscr{V}$ such that: (i) $\mathscr{V}$ is partially ordered by the relation $\supset$; (ii) if $V, W \in \mathscr{V}$ and $V \cap W \neq \Lambda$, then one has either $V \subset W$ or $V \supset W$ or $V = W$; (iii) for every $W \in \mathscr{V}$, the family of all $V \in \mathscr{V}$ containing W is well-ordered by the relation $\supset$.

2° Every set $V \in \mathscr{V}$ is a neighborhood of every point of V.

An R-space (E, τ) satisfies the axiom of separation T_1, iff for each $x \in E$ the intersection of all its neighborhoods $V \in \mathscr{V}$ reduces to x. We shall suppose that the R-space (E, τ) satisfies the axiom T_1.

As shown by P. Papić, every T_1, R-space is perfectly normal; moreover, the same author has given necessary and sufficient conditions for a T_1, R-space to be metrizable (cf. [5]). G. Kurepa (cf. [1]) proved that every T_1, R-space is of class $\mathscr{E}[M]$, M being a space of ordinal numbers; thus, by Theorem 3, every T_1, R-space admits at least one abstract distance f whose range is a space of ordinals. The function f has been defined by G. Kurepa as follows. Observe, first, that the local neighborhood base $\mathscr{V}x \subset \mathscr{V}$ of each point x of an R-space is well-ordered by $\supset$; then let $f(x, y)$ = ordinal number corresponding to the well-ordered intersection $\mathscr{V}x \cap \mathscr{V}y$, x and y being any two points in (E, τ). Let us remark that there is another abstract distance, reconstructing a given T_1, R-space, the range of which is also a space of ordinals; this was also described by G. Kurepa (cf. [6], p. 120).

Let us note finally that since it is perfectly normal, every T_1, R-space is completely regular; hence every T_1, R-space admits a metric over a topological semifield in the sense of the definitions given by M. Antonovski, V. Boltjanski and T. Sarimsakov in their book [4].

Is there any abstract distance still "nearer" to the usual metric than the metric (3) over topological semifields – for perfectly normal spaces? (This last class of spaces being still "nearer" to the class of metrizable topological spaces, in the usual sense.) Further reference and a more detailed survey of the results concerning abstract distances obtained by G. Kurepa, P. Papić and the author, can be found in the author's book [6], § 12.

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