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In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. [148]--151.

Persistent URL: http://dml.cz/dmlcz/700941

# ON THE TOPOLOGICAL PRODUCT OF DISCRETE $\lambda$-COMPACT SPACES

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In a previous paper [1] we investigated general intersection properties of abstract sets and applied our results to topological products of discrete spaces. In the present paper after restating some of our old results we solve one of the problems left open in [1] and state some old and new problems.

A topological space $\mathscr{X}$ is said to be $\kappa$-compact if every family $\mathscr{M}$ of closed subsets of it with void intersection contains a subfamily $\mathscr{M}' \subseteq \mathscr{M}$, $\overline{\overline{\mathscr{M}'}} < \aleph_\kappa$ with void intersection. 0-compactness means ordinary compactness (bicompactness in the sense of P. S. ALEXANDROFF and P. URYSON who introduced this terminology).

1-compact spaces are the Lindelöf spaces.

The symbol $\mathbf{T}(m, \lambda) \to \kappa$ will denote the following statement: If $\mathscr{F}$ is a family of discrete $\lambda$-compact topological spaces, $\overline{\overline{\mathscr{F}}} = m$, then the topological product of the elements of $\mathscr{F}$ is $\kappa$-compact. $\mathbf{T}(m, \lambda) \nrightarrow \kappa$ denotes the negation of the above statement.

Tychonov's classical theorem can be stated as $\mathbf{T}(m, 0) \to 0$ for every cardinal number $m$.

If we use the generalised continuum hypothesis a theorem of J. Łos can be stated as:

(1) $$\mathbf{T}(\aleph_{\alpha+1}, 1) \nrightarrow \alpha \quad \text{for every} \quad \alpha \geqq 1$$

provided $\aleph_\alpha$ is regular and of measure 0.[1]) (See [2].)

We proved

(2) $$\mathbf{T}(\aleph_{\alpha+n}, \alpha + 1) \nrightarrow \alpha + n$$

for every ordinal number $\alpha$ and for every $1 \leqq n < \omega$.

It is easy to see that (2) is best possible but it gives no information if $n \geqq \omega$. In [1] we proposed among others the following problems:

(3) $$\mathbf{T}(\aleph_\omega, 1) \to \omega \ ?$$

(4) $$\mathbf{T}(\aleph_{\omega+2}, 1) \to \omega + 2 \ ? \quad (\text{or } \mathbf{T}(\aleph_{\omega+1}, 1) \to \omega + 1 \ ?).$$

We can not answer (4) ($\mathbf{T}(\aleph_{\omega+2}, 1) \nrightarrow \omega + 1$ follows from (1)). But we shall prove that the answer to (3) is negative.

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[1]) The cardinal number $m$ is said to be of measure $o$, if every two valued $\sigma$-measure defined on all subsets of a set $S$, $\overline{\overline{S}} = m$ vanishes identically provided $M(\{x\}) = 0$ for every $x \in S$.

Before we give the (simple) proof we would like to state a few problems most of which have been already stated in [1]. First we need some definitions:

The family $\mathscr{F}$ of sets is said to possess property **B** if there exists a set $B$ such that

$$F \cap B \neq 0 \quad \text{and} \quad F \nsubseteq B \quad \text{for every} \quad F \in \mathscr{F} .$$

$\mathscr{F}$ is said to possess property $\mathbf{B}(s)$ if there is set $B$ such that

$$1 \leqq \overline{\overline{F \cap B}} < s \quad \text{for every} \quad F \in \mathscr{F} .$$

If $\overline{\overline{F}} = p$ for every $F \in \mathscr{F}$ we briefly write $p(\mathscr{F}) = p$. The family $\mathscr{F}$ is said to possess property $\mathbf{C}(\xi r)$ if $\overline{\overline{F_1 \cap F_2}} < r$ for every $F_1 \neq F_2 \in \mathscr{F}$.

Let $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ briefly denote the following statement:

Every family $\mathscr{F}, \overline{\overline{\mathscr{F}}} = m$, $p(\mathscr{F}) = p$ which possesses property $\mathbf{C}(r)$ possesses property $\mathbf{B}(s)$ too.

$\mathbf{M}(m, p, r) \nrightarrow \mathbf{B}(1)$ denotes then negation of this statement. (2) was deduced from the following result of [1]

(5) $\quad \mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \rightarrow \mathbf{B}((r-1)(n+1)+2), \quad \omega > r, n, \quad r \geqq 1, \quad n \geqq 0$

and

$$\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \nrightarrow \mathbf{B}((r-1)(n+1)+1), \quad \omega > r, n, \quad r \geqq 1, \quad n \geqq 0 .$$

(See Theorems 8 and 10 of [1].)

Namely it is easy to see that if there exists a family $\mathscr{F}, \overline{\overline{\mathscr{F}}} = \aleph_\beta$, $p(\mathscr{F}) = \aleph_\alpha$ and an integer $s$ such that every subfamily $\mathscr{F}'$ of power $< \aleph_\beta$ of $\mathscr{F}$ possesses property $\mathbf{B}(s)$ but $\mathscr{F}$ does not possess property $\mathbf{B}(s)$ then we have $\mathbf{T}(\aleph_\beta, \alpha + 1) \nrightarrow \beta$.

The results of [1] show that the investigation of the symbol $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ can not lead to the existence of such families for $\beta \geqq \alpha + \omega$.

Perhaps question of the following type can lead to new results in this direction.

Does there exist a family $\mathscr{F}$, $p(\mathscr{F}) = \aleph_\alpha$, $\overline{\overline{\mathscr{F}}} = \aleph_{\alpha+\gamma}$ and an integer valued function $l(F)$, defined for every $F \in \mathscr{F}$ satisfying the following conditions:

If $\mathscr{F}' \nsubseteq \mathscr{F}$, $\overline{\overline{\mathscr{F}'}} < \aleph_{\alpha+\gamma}$ then there exists a set $B'$ such that $1 \leqq \overline{\overline{F \cap B'}} < l(F)$ for every $F \in \mathscr{F}'$.

If $F \cap B \neq 0$ for every $F \in \mathscr{F}$ then $\overline{\overline{F \cap B}} \geqq l(F)$ for at least one $F \in \mathscr{F}$.

If $\alpha = 0$, $\gamma < \omega$ the existence of such a family follows from (5). The simplest cases where we do know the answer are $\alpha = 0$, $\gamma = \omega$ or $\gamma = \omega + 1$. It is obvious that a positive solution of this problem for $\alpha = 0$, $\beta = \omega + 1$ would fournish a proof of

$$\mathbf{T}(\aleph_{\omega+1}, 1) \nrightarrow \omega + 1 .$$

Here are some problems which would all have been consequences of $\mathbf{T}(\aleph_2, 1) \rightarrow 2$ (which we know is false). The answer to these questions is probably negative but we can not disprove them.

Let $\mathscr{F}$ be a family, $\overline{\overline{\mathscr{F}}} = \aleph_2$, $p(\mathscr{F}) = \aleph_0$ such that every $\mathscr{F}' \subseteq \mathscr{F}$, $\overline{\overline{\mathscr{F}'}} \leqq \aleph_1$ possesses property **B**. Does then $\mathscr{F}$ necessarily possess property **B** too?

A family $\mathscr{F}$ is said to have property **G** if there exists a function $f(F) \in F$ defined for every $F$ of $\mathscr{F}$ such that $f(F_1) \neq f(F_2)$ for $F_1 \neq F_2 \in \mathscr{F}$.

Let $\mathcal{F}$ be a family ($\overline{\overline{\mathcal{F}}} = \aleph_2$, $p(\mathcal{F}) = \aleph_0$) such that every $\mathcal{F}' \subseteq \mathcal{F}$, ($\overline{\overline{\mathcal{F}'}} \leqq \aleph_1$) possesses property **G**. Does then $\mathcal{F}$ necessarily possess property **G** too? This problem is due to W. Gustin.

Let there be given a graph $\mathcal{G}$ of power $\aleph_2$. Suppose that every subgraph of power $\leqq \aleph_1$ of it has chromatic number $\leqq \aleph_0$ (i. e. its vertices can be coloured with $\aleph_0$ colours so that two vertices of the same colour are never connected). Is it then true that $\mathcal{G}$ has chromatic number $\leqq \aleph_0$?

Let there be given a graph $\mathcal{G}$ of power $\aleph_2$. Suppose that for every subgraph $\mathcal{G}'$ of $\mathcal{G}$ of power $\leqq \aleph_1$ its edges can be directed in such a way that the number of edges emanating from an arbitrary vertex is finite. Is it true that the same holds for the graph $\mathcal{G}$? (This would not follow from $\mathbf{T}(\aleph_2, 1) \to 2$.)

Now we are going to outline the solution of problem (3).

Using the generalized continuum hypothesis we can prove the following theorem:

*Suppose that $\aleph_{\alpha+\gamma}$ is singular $cf(\gamma) \leqq \omega$ ($cf$ = cofinal) then*

$$\mathbf{T}(\aleph_{\alpha+\gamma}, \alpha + 1) \not\to \alpha + \gamma .$$

We give the proof of the simplest case $\alpha = 0$, $\gamma = \omega$, $(cf(\gamma) = 0)$.

By (2) for every $\omega > n \geqq 1$ there exists a family $\mathcal{F}_n$, $\overline{\overline{\mathcal{F}}}_n = \aleph_n$, $p(\mathcal{F}) = \aleph_0$ such that the topological product $\aleph_n$ of the discrete spaces $F \in \mathcal{F}_n$ is not $n$-compact, i. e. there exists a family $\mathcal{M}_n$ of closed subsets of $X_n$ such that $\mathcal{M}_n$ has a void intersection and $\mathcal{M}' \subseteq \mathcal{M}$, $\overline{\overline{\mathcal{M}'}} < \aleph_n$ implies that $\mathcal{M}'$ has a non-void intersection.

Let $J = \{1, \ldots, n, \ldots\}$ be the set of integers and consider the space $\mathcal{X} = J \times J_1 \times \ldots \times J_n \times \ldots$.

Let $\mathcal{M}_n^*$ be the following system of subsets of $\mathcal{X}$: $U \in \mathcal{M}_n^*$ if and only if there exists an $U' \in \mathcal{M}_n$ such that $U = J \times \mathcal{X}_1 \times \ldots \times \mathcal{X}_{n-1} \times U' \times \mathcal{X}_{n+1} \times \ldots \cup \cup J_n \times \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \times \ldots$ where $J_n \subseteq J$, $J_n = \{n, n+1, \ldots\}$. It is obvious that $\mathcal{M}_n^*$ consists of closed subsets of $\mathcal{X}$. Put $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n^*$. Considering that the $\mathcal{M}_n$'s have a void intersection it follows that $\mathcal{M}$ has a void intersection and it is easy to verify that $\mathcal{M}' \subseteq \mathcal{M}$, $\overline{\overline{\mathcal{M}'}} < \aleph_\omega$ implies that $\mathcal{M}'$ has a non-void intersection. Hence $\mathcal{X}$ is the topological product of $\aleph_\omega$ discrete 1-compact spaces and is not $\omega$-compact.

In the general case the proof becomes a little more complicated since instead of $J$ and the sets $J_n$ we have to use a system which proves $\mathbf{T}(\aleph_{cf(\gamma)}, \alpha + 1) \not\to cf(\gamma)$ and as to the existence of the systems corresponding to the $\mathcal{M}_n$'s we have to refer to Łos's theorem.

For singular $\aleph_{\alpha+\gamma}$ the simplest unsolved problem is the following:

$$\mathbf{T}(\aleph_{\omega_{\omega+1}}, 1) \to \omega_{\omega+1} ?$$

Our proof for $\mathbf{T}(\aleph_{\omega_{\omega+1}}, 1) \not\to \omega_{\omega+1}$ breaks down since we can not even prove

$$\mathbf{T}(\aleph_{\omega+1}, 1) \not\to \omega + 1$$

**References**

[1] *P. Erdös* and *A. Hajnal*: On a property of families of sets. Acta Math. Acad. Sci. Hung. *12* (1961), 87–123.
[2] *J. Łos*: Linear equations and pure subgroups. Bull. Acad. Polon. Sci. Math. *7* (1959), 13–18.