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PERIODIC HOMEOMORPHISMS OF THE 3-SPHERE

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Let M be a triangulated 3-sphere, and let f be a periodic simplicial homeomorphism of M onto itself. Suppose that f preserves orientation and has a fixed point; let F be the fixed-point set of f; and let n be the period of f. By methods and results of P. A. SMITH [1] it follows that F is always a (simple closed) polygon. A well-known conjecture due to Smith (discussed by S. EILENBERG [2] in his 1949 report on the problems of topology) asserts that F is never knotted.

It has been shown by D. MONTGOMERY and H. SAMELSON [3] that for $n = 2$, F cannot be a simplicial standard torus knot. They showed also that if $n = 2$ and F is unknotted, then f is topologically equivalent to a rotation.

The main result reported here is that the second of these results holds without restriction $n = 2$:

Theorem 1. *If $f : M \leftrightarrow M$ is periodic and simplicial, and preserves orientation, and F is unknotted, then f is topologically equivalent to a rotation.*

This result is derived from the following purely homological theorem:

Theorem 2. *Let M be a triangulation, not necessarily of the 3-sphere, but of a compact 3-manifold having the homology groups of the 3-sphere. Let $f : M \leftrightarrow M$ be a simplicial homeomorphism of M onto itself, preserving orientation, with period n and having the polygon F as its fixed-point set. Then there is a polyhedral disk D_1 with handles such that (1) F is the boundary of D_1 and (2) each two different sets $f^i(D_1), f^j(D_1)$ intersect only in F.*

Here by a disk with handles we mean a compact, connected orientable 2-manifold whose boundary is a single polygon. If D_1 is an actual disk (with no handles), then for the case $M = S^3$ it follows that each pair of geometrically adjacent disks

$$D_i = f^{i-1}(D_1), \quad D_j = f^{j-1}(D_1)$$

form the boundary of a 3-cell, say C_i; and the 3-cells C_i are cyclically permuted by f. From this it follows that f is topologically equivalent to a rotation. In fact, to deduce Theorem 1 from Theorem 2, we show that if F is the boundary of *some* polyhedral disk D, and the genus of D_1 is positive, then this genus can always be reduced.

These results will appear soon with proofs in a paper in the Illinois Journal of

Mathematics. The proofs are by explicit geometric construction, and do not lend themselves to informal summary. Since the difference in date of publication will be small in any case, we do not attempt to give such a summary here.

Bibliography

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[3] *D. Montgomery* and *H. Samelson*: A theorem on fixed points of involutions in S^3. Canad. J. Math., *7* (1955), 208–220.