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In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. [307]--310.

Persistent URL: http://dml.cz/dmlcz/700934

# PROBABILITY MEASURES ON NON-COMMUTATIVE SEMIGROUPS

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Let $S$ be a compact semigroup, i. e. a compact Hausdorff space with a jointly continuous binary operation (multiplication) under which it forms a semigroup. A probability measure on $S$ is a real-valued, non-negative, countably additive, regular set function $\mu$ defined on the Borel subsets of $S$ such that $\mu(S) = 1$. The totality of all such measures will be denoted by $\mathfrak{M}(S)$.

Let $\omega(S)$ be the Banach space of real continuous functions on $S$. Recall the $1-1$ correspondence between the measure $\mu \in \mathfrak{M}(S)$ and the continuous linear functionals $\Phi$ on $\omega(S)$ with the properties $f \geqq 0 \Rightarrow \Phi(f) \geqq 0$ and $\Phi(1) = 1$ that is given by the relation $\Phi(f) = \int f(x)\,\mathrm{d}\mu(x)$.

We may (and we shall) consider $\mathfrak{M}(S)$ embedded in $\omega(S)^*$ (the first conjugate space of $\omega(S)$) and we regard also $S$ as embedded (in an obvious way) in $\mathfrak{M}(S)$.

We give $\omega(S)^*$ the weak*-topology (so that $\mu_\alpha \to \mu$ means $\int f\,\mathrm{d}\mu_\alpha(x) \to \int f\,\mathrm{d}\mu(x)$ for $f \in \omega(S)$) and define the product $\mu\nu$ by means of $\int f(x)\,\mathrm{d}(\mu\nu)(x) = \iint f(yz)\,\mathrm{d}\mu(y)\,.$ $.\,\mathrm{d}\nu(z)$, $f \in \omega(S)$. The set $\mathfrak{M}(S)$ becomes a compact semigroup.

If for $\mu \in \mathfrak{M}(S)$ $C(\mu)$ denotes the support of $\mu$, it is known that $C(\mu\nu) = C(\mu)\,.$ $.\,C(\nu)$.

In the last years the structure of $\mathfrak{M}(S)$ has been studied by several authors, in particular by Н. Н. Воробьев [9], J. G. Wendel [10], E. Hewitt-H. S. Zuckerman [2], Б. М. Клосс [3], I. Glicksberg [7], K. Stromberg [8]. But these authors consider only the case of groups and abelian semigroups. The essential novelty of our contribution is that we are going beyond the restriction of commutativity even in the non-group case (for $S$). We give here some results, the detailed proofs of which will be published in the Czechoslovak Mathematical Journal ([6] and [7]).

**1.** The first problem is to identify the *idempotents* $\in \mathfrak{M}(S)$.

A subset $L$ of any semigroup $T$ is called a left ideal of $T$ if $TL \subset L$ holds. Right and two-sided ideals are defined analogously. A semigroup $T$ is called simple (more precisely simple without zero) if it does not contain a two-sided ideal $\neq T$. If $T$ is a compact simple semigroup it is known that $T$ contains minimal left and right ideals. In fact $T = \bigcup_\alpha R_\alpha = \bigcup_\beta L_\beta$, where $R_\alpha[L_\beta]$ runs through all (disjoint) minimal right [left] ideals of $T$. Also $R_\alpha \cap L_\beta = R_\alpha L_\beta = G_{\alpha\beta}$ is a closed (compact) group and $T$ can be written as a union of closed (hence compact) topologically isomorphic groups: $T = \bigcup_{\alpha} \bigcup_{\beta} G_{\alpha\beta}$.

Let now be $\varepsilon$ an idempotent $\in \mathfrak{M}(S)$. Б. М. Клосс [3] proved that $C(\varepsilon)$ is a closed (hence compact) simple subsemigroup of $S$.

In what follows we shall suppose that $S$ contains only a *finite* number of idempotents though some of the results are valid under more general suppositions or even without any assumption of this kind.

If $\varepsilon$ is an idempotent $\in \mathfrak{M}(S)$ and $C(\varepsilon) = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$ is the group-decomposition of $C(\varepsilon)$, it can be proved that $\varepsilon$ restricted to $G_{\alpha\beta}$ is an invariant measure on $G_{\alpha\beta}$. This implies: If $\mu_{\alpha\beta}$ is the normalized Haar measure on $G_{\alpha\beta}$ (extended in an obvious way to $S$) then $\varepsilon = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu_{\alpha\beta}$, $\xi_\alpha, \eta_\beta$ being positive numbers satisfying the relation $\sum_{\alpha=1}^{s} \xi_\alpha = \sum_{\beta=1}^{r} \eta_\beta = 1$.

Conversely, if $H = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$ is *any* closed simple subsemigroup of $S$, there exists at least one idempotent having $H$ for its support and every idempotent $\in \mathfrak{M}(S)$ with the support $H$ is of the form $\sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu_{\alpha\beta}$, where $\sum_{\alpha=1}^{s} \xi_\alpha = \sum_{\beta=1}^{r} \eta_\beta = 1$, $\xi_\alpha \eta_\beta > 0$.

The study of the idempotents $\in \mathfrak{M}(S)$ is simplified by the fact that the location of all simple subsemigroups of $S$ can be clarified to some extent and we may restrict the attention to the "maximal simple subsemigroups of $S$" (which in general need not have an empty intersection). A useful lemma is the following statement: Every closed subsemigroup of a compact simple semigroup (without zero) is itself simple (see [5]).

**2.** In some problems (see f. i. [4]) the *primitive* idempotents $\in \mathfrak{M}(S)$ are of decisive importance. An idempotent $f$ of any semigroup $T$ is said to be primitive if there does not exist an idempotent $e \in T$, $e \neq f$, such that $fe = ef = e$ holds. The following results clarify the structure of the set of all primitive idempotents $\in \mathfrak{M}(S)$.

Denote by $N$ the kernel of $S$ (i. e. the minimal two-sided ideal of $S$) and by $\mathfrak{N}$ the kernel of $\mathfrak{M}(S)$. If $N = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$ is the group-decomposition of $N$, then all primitive idempotents $\in \mathfrak{M}(S)$ are given by the expression $\sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu_{\alpha\beta}$, where $\xi_\alpha$, $\eta_\beta$ are *non-negative* numbers satisfying $\sum_{\alpha=1}^{s} \xi_\alpha = \sum_{\beta=1}^{r} \eta_\beta = 1$. The kernel $\mathfrak{N}$ is identical with the set of all primitive idempotents $\in \mathfrak{M}(S)$. Consider finally the set $\mathfrak{T}$ of all $(s + r)$-tuples of non-negative real numbers $(\xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_r)$ with $\sum_{\alpha=1}^{s} \xi_\alpha = \sum_{\beta=1}^{r} \eta_\beta = 1$ and define in $\mathfrak{T}$ a multiplication $\circ$ by

$$(\xi_1', \ldots, \xi_s', \eta_1', \ldots, \eta_r') \circ (\xi_1'', \ldots, \xi_s'', \eta_1'', \ldots, \eta_r'') = (\xi_1', \ldots, \xi_s', \eta_1'', \ldots, \eta_r'') .$$

Then $\mathfrak{N}$ is isomorphic with $\mathfrak{T}$.

**3.** Another important problem is to identify the *maximal groups* contained in $\mathfrak{M}(S)$.

Let $\varepsilon = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu_{\alpha\beta}$ (with fixed chosen $\xi_\alpha$, $\eta_\beta$) be an idempotent $\in \mathfrak{M}(S)$ and $\mathfrak{G}(\varepsilon)$ the (uniquely determined) maximal group of $\mathfrak{M}(S)$ containing $\varepsilon$ as its unity element. Denote $C(\varepsilon) = H$. There exists always a uniquely determined maximal simple subsemigroup $H' \supset H$ of $S$ containing exactly the same idempotents as $H$. Construct the double coset decomposition

$$H' = H \cup HaH \cup HbH \cup \ldots , \quad (a, b, \ldots \in H')$$

with disjoint summands. This is possible (see [5]). If $H = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$, $H' = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}'$, $G_{\alpha\beta} \subset G_{\alpha\beta}'$, there exists also a decomposition of each $G_{\alpha\beta}'$ of the form

$$G_{\alpha\beta}' = G_{\alpha\beta} \cup G_{\alpha\beta} a G_{\alpha\beta} \cup G_{\alpha\beta} b G_{\alpha\beta} \cup \ldots$$

(In fact $HaH \cap G_{\alpha\beta}' = G_{\alpha\beta} a G_{\alpha\beta}$.) With these notations the following results hold:

a) If $\mu \in \mathfrak{G}(\varepsilon)$, we have $C(\mu) = HaH$ with a suitably chosen $a \in H'$.

b) There is a unique element $\mu \in \mathfrak{G}(\varepsilon)$ with $C(\mu) = HaH$ and $\mu$ is exactly the element $\mu = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \tau_{\alpha\beta}$, where $\tau_{\alpha\beta} = \mu_{\alpha\beta} a \mu_{\alpha\beta}$.

c) A class $HaH$ is the support of an element $\in \mathfrak{G}(\varepsilon)$ if and only if $G_{\alpha\beta} a G_{\alpha\beta}$ is contained in the normalizer $G_{\alpha\beta}^{(0)}$ of $G_{\alpha\beta}$ in $G_{\alpha\beta}'$. (If the last statement holds for one couple $(\alpha, \beta)$, then it holds for all couples $(\sigma, \varrho)$, $\sigma = 1, \ldots, s$; $\varrho = 1, \ldots, r$.)

d) The group $\mathfrak{G}(\varepsilon)$ is algebraically isomorphic to the quotient group $G_{\alpha\beta}^{(0)}/G_{\alpha\beta}$.

**4.** The following limit theorems are consequences of the results stated above:

Consider the sequence $\{\mu, \mu^2, \mu^3, \ldots\}$. If $\lim_{n=\infty} \mu^n$ exists, it is equal to the unique idempotent $\varepsilon$ contained in the closure of $\{\mu, \mu^2, \mu^3, \ldots\}$. This is the case if and only if $\mu\varepsilon = \varepsilon\mu = \varepsilon$.

A rather algebraic characterisation of the existence of the limit considered is the fulfilment of the relation $H\, C(\mu)\, H = H$, where $H = C(\varepsilon)$.

The next result holds even without the assumption concerning the finiteness of the number of idempotents $\in S$. Consider the sequence $\sigma_n = 1/n\ (\mu + \mu^2 + \ldots + \mu^n)$, $n = 1, 2, 3, \ldots$ Then $\lim_{n=\infty} \sigma_n$ exists always and it is equal to an idempotent $\sigma \in \mathfrak{M}(S)$. If $P$ is the closure of the algebraic semigroup generated by $C(\mu)$ and $J$ is the minimal two-sided ideal of $P$, then we have $C(\sigma) = J$.

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