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TOPOLOGICAL ASPECTS OF CONFORMAL MAPPING THEORY

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The purpose of this note is not to present new results but to review some topological aspects of old ones.

A conformal map h of a domain D of the extended complex plane onto a set hD of the same plane carries topological properties of D over to hD — for example, hD is also a domain. In particular, when h is univalent, or schlicht, D and hD are homeomorphic as subspaces of the complex plane or as two-dimensional manifolds and therefore have the same connectivity. The fundamental investigations of S. STOÏLOW [5] showed that from a topological point of view the conformal maps of such manifolds on one another are characterized as light interior mappings.

At the same time the map h induces certain relations between the boundaries of D and hD. For example, when h is univalent and hD is an open circular disc, it is well known that h has a unique continuous extension to the closure of D which maps it onto the closed disc, and that the extended map attaches to each point on the circumference an antecedent set which is a prime end of D. In particular, when the boundary of D is a simple closed arc or Jordan curve, each prime end reduces to a single point and the extended map is a homeomorphism. In general h can be extended from D to its Čech-Stone compactification βD so as to map the latter continuously onto the closure of hD; and the relations between the boundaries of D and hD should be studied in terms of this extension. Indeed, the problem thus posed may be generalized as a purely topological problem: if S is a locally compact Hausdorff subspace of a topological space T and h a continuous map of S into a second topological space, to study in terms of βS the relations induced by h between the boundaries of S and hS, thus generalizing the classical theory of prime ends. The papers contributed by S. EILENBERG and K. KURATOWSKI to this Symposium deal with some aspects of this problem.

As S. STOÏLOW [5] has pointed out, a conformal map h induces a natural correspondence between the boundary components of D and those of hD. When h is univalent, so also is this correspondence. The boundary components are of two types — those consisting of a single point and those which are proper continua. Under a univalent conformal map corresponding boundary components are not necessarily of the same type. Following L. SARIO [4], we call a boundary component stable if its type persists under all such maps, and unstable otherwise. For example, every isolated boundary component is stable, and unstable boundary components can occur only for domains of infinite connectivity. A stable boundary component is called weak or strong according as it is of the first or the second type. This classification has been rather extensively but still inconclusively investigated in recent years by L. Sario [4], K. OIKAWA [1], and others. It has an intimate connection with the problem of mapping a given domain conformally and univalently onto a domain of some standard type.

Indeed, let us consider here two of the best known standard types — the circularly slit disc and the radially slit disc. A connected open subset of an open circular disc is called a circularly slit disc if it contains the center of the disc and has as its boundary components the circumference of the disc and a finite or infinite number of concentric circular arcs or slits (some of which may reduce to single points). Similarly, a connected open subset of an open circular disc is called a radially slit disc if it contains the center of the disc and has as its boundary components the circumference of the disc and a finite or infinite number of radial segments or slits (some of which may reduce to single points). In addition, it is necessary to take into account a somewhat more general type of domain, the radially slashed disc — that is to say, an open connected subset of an open circular disc containing the center of the disc and having as its boundary components a finite or infinite number of radial slits and a single connected set of zero measure obtained by joining to the circumference of the disc a finite or infinite number of radial segments. Indeed it is also convenient to replace in these descriptions the open circular disc and its circumference by the finite plane and the point at infinity, thus obtaining domains which may be called circularly slit discs of infinite radius, radially slit discs of infinite radius, and radially slashed discs of infinite radius, respectively.

The existence of univalent conformal maps of a given domain D upon a domain of one or another of these types is closely associated with the extremization of the functional

$$Q_t : h \to L_t(h)^2/A(h)$$

where h is a function analytic in D, t is a prescribed point of D, $L_t(h)$ is the linear magnification effected by h at t, and $A(h)$ is the area of hD. Thus

$$L_t(h) = |h'(t)|\,, \quad A(h) = \iint\limits_D |h'(z)|^2 \,\mathrm{d}o\,.$$

Since Q_t is homogeneous of zero degree in h, the extrema are the same as those obtained by fixing the value of $L_t(h)$ and extremizing $A(h)$, or vice versa. For the same reason there is no loss of generality in normalizing h in one way or another – for example, by putting $h(t) = 0$, $h'(t) > 0$, and fixing $h'(t)$ or $A(h)$. It is well known [6] that in terms of the Hermitian semi-norm $A(h)^{\frac{1}{2}}$ the functional Q_t is weakly upper semi-continuous. The set of competing functions of greatest interest in the present context is the set of those univalent functions which map D into a circular disc with center 0 in such a way that the boundary component of hD corresponding to a prescribed boundary component Γ of D is the circumference of the disc. Unless Γ is weak, this set is non-empty. The greatest lower bound of Q_t over this set will be denoted as α_t, its least upper bound as β_t. It is convenient to introduce also the quantities $r_t = 1/\sqrt{(\pi\beta_t)}$ and $R_t = 1/\sqrt{(\pi\alpha_t)}$.

The extremal problem for a general domain cannot be solved directly but has to be discussed by combining approximation techniques with certain a priori estimates of $Q_t(h)$ for special domains – such as the Bieberbach inequality for the circular disc and the stronger, more sophisticated inequalities given by Reich and Warschawski for circularly slit and radially slit discs [2, 3]. The first step is to solve the problem for domains of finite connectivity. If Γ is not weak, there exist unique normalized competing functions at which Q_t attains its extrema. The function h such that $Q_t(h) = \alpha_t$ maps D onto a radially slit disc which has radius R_t for the normalization $h'(t) = 1$. Similarly the function h such that $Q_t(h) = \beta_t$ maps D onto a circularly slit disc which has radius r_t for the same normalization. The problem for a general domain is then treated by studying exhaustions of D by subdomains of finite connectivity with distinguished boundary components converging in the sense of Stoïlow to D. These approximating subdomains are chosen so as to have no weak boundary components. The limit functions obtained from the extremizing functions for the approximating subdomains are then candidates for mapping D on some domain of standard type, and possibly even for extremizing Q_t.

Since Q_t is weakly upper semi-continuous the problem of maximizing this functional can be successfully discussed by the procedure described above [2, 6]. If Γ is not weak, there is a unique maximizing function which maps D onto a circularly slit disc with circumference corresponding to Γ and with radius r_t when the normalization is given by the condition $h'(t) = 1$. Furthermore the case where Γ is weak is characterized by the equivalent conditions $\beta_t = 0$, $r_t = +\infty$. To test whether Γ is weak, it suffices to observe that r_t can be computed as a limit of radii associated with the approximating subdomains [1, 4]. It may be conjectured that in this case D can be mapped univalently and conformally on a circularly slit disc of infinite radius.

The minimization problem is more difficult to analyze and still remains comparatively obscure. It is not known, for instance, whether D can always be mapped on a radially slit disc when Γ is not weak or even when Γ is strong. However, recent results of E. Reich [3] show that the exhaustion procedure introduces a quantity ϱ_t (which he denotes otherwise) as a limit of radii associated with the approximating subdomains, $R_t \leqq \varrho_t \leqq +\infty$, and that when ϱ_t is finite D can be mapped on a radially slashed disc with radius ϱ_t by a function satisfying the condition $h'(t) = 1$. Furthermore, it is known [1, 4] that $\varrho_t < +\infty$ is sufficient for Γ to be strong. It may be conjectured that in the case $\varrho_t = +\infty$ the domain D can be mapped univalently and conformally on a radially slashed disc of infinite radius when Γ is strong and on a radially slit disc of infinite radius when Γ is weak; and perhaps even that these two possibilities can be characterized by the conditions $\alpha_t > 0$ and $\alpha_t = 0$ respectively – though there is little evidence to this effect.

References

[1] *K. Oikawa*: On the Stability of Boundary Components. Pacific Journal of Mathematics, *10* (1960), 263–294.

[2] *Edgar Reich* and *S. E. Warschawski*: On Canonical Conformal Maps of Regions of Arbitrary Connectivity. Pacific Journal of Mathematics, *10* (1960), 965–985.

[3] *Edgar Reich*: On Radial Slit Mappings (to appear).

[4] *L. Sario*: Strong and Weak Boundary Components. Journal d'Analyse Mathématique, *5* (1956–57), 389–398.

[5] *S. Stoïlow*: Leçons sur les principes topologiques de la théorie des fonctions analytiques. Paris (1938), X + 148 pages.

[6] *M. H. Stone*: Hilbert Space Methods in Conformal Mapping. Proceedings of the International Symposium on Linear Spaces 1960, Jerusalem (1961), 409–425.