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REMOTE POINTS IN βR

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The results presented here were obtained jointly with N. J. FINE.[1])

It seems natural to conjecture that each point p in βR is a limit point of a suitable increasing or decreasing sequence in R – that is, that $p \in \mathrm{cl}_{\beta R} D$ for some discrete subset D of R that is closed in R. (As usual, βX denotes the Stone-Čech compactification of X, and R denotes the real line.) However, this is false. In fact (as was pointed out by W. F. EBERLEIN several years ago), it is easy to find a point in βR that is not a limit point of any closed subset of R that has finite Lebesgue measure.

Our present question is: does there exist a point in βR that is not in the closure of any discrete subset of R (closed or not). The problem seems surprisingly difficult. If we assume the continuum hypothesis [CH], then we can find such a "remote" point. (But we do not know whether [CH] is necessary.) An equivalent formulation in terms of subsets of R itself is: [CH] there exists a "z-ultrafilter" on R no member of which is nowhere dense. (A z-ultrafilter is a maximal family of zero-sets with the finite intersection property. On the real line, zero-sets – i. e., sets of zeros of continuous real-valued functions – are the same as closed sets.)

More generally, we have:

Theorem. *Let X be a completely regular (Hausdorff) space such that* (a) *the set of all isolated points does not admit a Ulam* 2*-valued measure, and* (b) *there exists a family $\mathscr{U}$ of $\aleph_1$ dense open sets such that every dense open set in X contains a member of $\mathscr{U}$. Then X admits an unbounded continuous real-valued function if and only if there exists a z-ultrafilter $\mathscr{A}$ on X such that $\bigcap \mathscr{A}$ is void and every dense open set contains a member of $\mathscr{A}$.*

Hypothesis (a) holds, in particular, if the cardinal of the set of isolated points of X is smaller than the first strongly inaccessible cardinal. In the proof of necessity, this hypothesis is not used at all. (Other portions of the theorem may also be stated more generally.)

Our result about remote points in βR follows easily from [CH] and the theorem. If we consider the subspace consisting of the rationals and one remote point, we get:

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[1]) The complete article appears in the Proceedings of the American Mathematical Society, *13* (1962), 29—36.

Corollary. [CH] *There exists a countable, completely regular space without isolated points, one of whose points is not a limit point of any discrete set.*

Finally, it can be shown [CH] that the set of all remote points is actually dense in $\beta R - R$.