Miroslav Katětov

On a category of spaces

In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. 226--229.

Persistent URL: http://dml.cz/dmlcz/700924

ON A CATEGORY OF SPACES

M. KATĚTOV

Praha

If X is a set, let $E(X)$ denote the linear space of all "formal linear combinations" $\Sigma\lambda_i x_i$ where λ_i are numbers (real or complex), $x_i \in X$. A topology λ on $E(X)$ under which $E(X)$ is a locally convex topological linear space (abbreviated: LC-space, LC-topology) will be called a Λ-structure on X, and the pair (X, λ) will be called a Λ-space. If (X_1, λ_1), (X_2, λ_2) are Λ-spaces, then a mapping φ of X_1 into X_2 will be called Λ-continuous (or a Λ-mapping or simply a morphism) if the linear extension of φ is a continuous (with respect to λ_1, λ_2) mapping of $E(X_1)$ into $E(X_2)$. If λ_1, λ_2 are Λ-structures on X, then λ_1 is finer than λ_2 (λ_2 is coarser than λ_1) if the identity mapping of (X, λ_1) onto (X, λ_2) is Λ-continuous.

In the following, definitions and propositions (as well as some examples) are given in the form of short remarks, some of them formulated somewhat loosely. A full exposition is to be submitted to the *Czechoslovak Mathematical Journal.*

1. Several years ago, V. EFREMOVIČ suggested the investigation of properties of metric spaces invariant under mappings satisfying (in both directions) the Lipschitz condition. This suggestion may be carried out as follows. Two metrics ϱ_1, ϱ_2 on X are called L-equivalent if, for some positive α, β,

$$\alpha\,\varrho_1(x, y) \leqq \varrho_2(x, y) \leqq \beta\,\varrho_1(x, y) \quad \text{for all} \quad x, y \in X\ ;$$

a set consisting of all L-equivalent metrics is called a quasi-metric; a pair (X, q), q being a quasi-metric on X, is called an L-space. Some simple properties of L-spaces have been examined by the present author (to appear in the Proceedings of the *Conference on Functional Analysis*, Warsaw 1960). Since every metric on X may by extended [1] to a norm on $E(X)$, L-spaces may be considered as a special case of Λ-spaces.

2. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be categories. If $S : \mathscr{A} \to \mathscr{C}$, $T : \mathscr{B} \to \mathscr{C}$ are covariant functors, then a category may be obtained in the following way: objects (morphisms) are pairs (a, b), (α, β) of objects (morphisms) from $\mathscr{A}$, $\mathscr{B}$ satisfying the conditions $Sa = Tb$, $S\alpha = T\beta$. If $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are respectively the categories of all sets, all LC-spaces, all linear spaces, and $S : \mathscr{A} \to \mathscr{C}$, $T : \mathscr{B} \to \mathscr{C}$ are natural functors ($SX = E(X)$, TY is the underlying linear space), then we obtain, essentially, the category of all Λ-spaces. Instead of the "fusing" of categories, as just described, a kind of "transfer of structures" may be considered. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be categories, the objects of $\mathscr{C}$ being those of $\mathscr{B}$

endowed with certain structures. If $S : \mathscr{A} \to \mathscr{B}$ is a covariant functor, we may consider a category whose objects are pairs (a, μ), a being an object from $\mathscr{A}$, μ a structure on Sa, and morphisms corresponding precisely to those morphisms $\alpha : a \to a'$ of $\mathscr{A}$ for which $S\alpha$ is a morphism for $\mathscr{C}$ mapping (Sa, μ) to (Sa', μ').

3. For any Λ-space X, denote by $\hat{E}(X)$ the completion of $E(X)$, and by $E'(X)$ the linear space (topologized in a suitable way) of continuous linear forms on $E(X)$.

As already mentioned, a metric on a set X induces, in a way described in [1], a norm, hence a topology on $E(X)$, and therefore a Λ-structure on X (of course, two L-equivalent metrics induce, in this sense, the same Λ-structure). Some trivial examples: if X is discrete, $\varrho(x, y) = 1$ for $x \neq y$, then

$$\hat{E}(X) \cong l, \quad E'(X) \cong m ;$$

if X is an interval, then $\hat{E}(X) \cong L$, $E'(X) \cong M$ ($\cong$ means isometric isomorphism).

Another class of Λ-spaces may be obtained as follows. A linear system Φ of real functions on X induces, under certain conditions, a weak LC-topology on $E(X)$; along with it, we may consider the corresponding "strong" topology, or some intermediate topologies. In particular, r-differentiable ($r = 1, 2, \ldots, \infty$) manifolds may be treated in this way; with $r = \infty$ and an appropriate topology for $E(X)$, $\hat{E}(X)$ consists, essentially, of distributions on X.

4. We shall use, for convenience, the expressions u-structure, δ-structure, t-structure to denote, respectively, a uniform structure, a proximity structure, a topology; the same convention is adopted, of course, for spaces. The letters u, δ, t will be used as generic symbols for a structure of the corresponding kind, and the letter c will stand for u, δ or t (in both meanings). The letter λ will be used as a generic symbol for a Λ-structure.

Every λ on X induces, in a natural way, for $c = u, \delta, t$, a c-structure on X denoted by $c(\lambda)$. If γ is a c-structure, then we denote by $\Lambda(\gamma)$ the set of all λ for which $c(\lambda) = \gamma$.

A Λ-space $X = (X, \lambda)$ as well as its structure λ will be called, respectively, weak, bounded, totally bounded if the topology of $E(X)$ is weak, X is bounded, is totally bounded as a subset of $E(X)$.

The following assertions are, for the most part, paraphrases of well known theorems:

(a) $\Lambda(u)$ always contains a finest structure;

(b) $\Lambda(u)$ does not necessarily contain a weak structure;

(c) $\Lambda(\delta)$ contains, in general, no finest structure (example: $N \times N$ imbedded in $\beta N \times \beta N$);

(d) $\Lambda(t)$ always contains a finest structure as well as a totally bounded weak structure (induced by the set of all bounded continuous functions);

(e) $\gamma = u, \delta$ is bounded (totally bounded) if and only if $\Lambda(\gamma)$ contains a bounded (totally bounded) λ (for the definition of boundedness and total boundedness of γ see [3], [4]).

5. Let $\mathscr{A}$, $\mathscr{B}$ be categories. We shall say that $\mathscr{B}$ is a retract of $\mathscr{A}$ if there exist covariant functors $S : \mathscr{A} \to \mathscr{B}$, $T : \mathscr{B} \to \mathscr{A}$ such that $S \circ T$ is the identity functor. It is easy to see that, for $c = u, \delta, t$, the category of all c-spaces is a retract of that of all Λ-spaces. Indeed, for $c = u$, we may put $T(X, u)$ equal to the Λ-space with the finest λ inducing u. As for $c = \delta, t$, it is only a matter of re-wording well known results to prove that the category of δ-spaces is a retract of that of u-spaces (observe that $T(X, \delta) = (X, u)$ with the coarsest u such that $\delta(u) = \delta$), and so on.

6. Let (X_i, ϱ_i), $i = 1, 2$, be metric spaces. The following definition of the direct product of the corresponding Λ-spaces (X_i, λ_i) appears, at the first sight, to be natural and convenient:

$$(X_1, \lambda_1) \times (X_2, \lambda) = (X_1 \times X_2, \lambda)$$

with λ induced by the metric

$$\varrho(x, y) = \varrho_1(x_1, y_1) + \varrho_2(x_2, y_2)\,.$$

Unfortunately, it may happen that, for some $f_i \in E'(X_i)$, the linear form $f_1 \otimes f_2$ on

$$E(X_1) \otimes E(X_2) = E(X_1 \times X_2)$$

is not continuous (this corresponds to the trivial fact that the product of two Lipschitz functions is not necessarily Lipschitz).

Therefore, it is necessary to look for more appropriate definitions of $X_1 \times X_2$. The following one seems to be convenient. Let (X_i, λ_i) be Λ-spaces. The topology of

$$E(X_1 \times X_2) = E(X_1) \otimes E(X_2)$$

will be determined by the following condition: a linear mapping φ of $E(X_1) \otimes \otimes E(X_2)$ into a normed linear space is continuous if and only if the corresponding bilinear mapping $\varphi(\xi_1, \xi_2)$, $\xi_i \in E(X_i)$, is equicontinuous in ξ_1 (ξ_2) for every bounded subset of X_2 (X_1). It is to be noted, however, that, under this definition, the product of L-spaces is not necessarily an L-space; it seems therefore appropriate to assign to a non-bounded metric space X instead of an L-space (as described at the beginning of the present note) a Λ-space defined as follows: a linear mapping of $E(X)$ into a normed linear space is continuous if and only if it is Lipschitz on every bounded set $A \subset X$.

Finally, we may add that the topology for $E(X_1) \otimes E(X_2)$ described in the above definition of the product of two Λ-spaces is, in general, intermediate between the topologies of the inductive and projective products of $E(X_1)$ and $E(X_2)$ introduced by A. Grothendieck.

References

[1] *R. Arens* and *J. Eells*: On embedding uniform and topological spaces. Pacif. J. Math. *6* (1956), 397–403.

[2] *A. Grothendieck*: Produits tensoriels topologiques et espaces nucléaires. Memoirs of the American Math. Soc. *16* (1955), 1–185.

[3] *J. Hejcman*: The boundedness in uniform spaces and topological groups. Czechosl. Math. J. *9* (84) (1959), 544–563.

[4] *M. Katětov*: Über die Berührungsräume. Wiss. Z. Humboldt-Univ. Berlin, Math.-Nat. R. *9* (1959-60), 685–691.