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Praha

A convergence space or $\mathfrak{L}$-space L is a space, in which the sequential topology is defined by means of a convergence. By convergence $\mathscr{L}$ we understand a system $\mathscr{L}$ of sequences $\{x_n\} \in \mathscr{L}$ of points $x_n \in L$ converging to certain points called limits and designated by the symbol $\lim x_n$ and fulfilling two Fréchet's axioms:

1. If $x_n = x$ for each $n = 1, 2, \ldots$, then $\{x_n\} \in \mathscr{L}$ and $\lim x_n = x$.
2. If $\lim x_n = x$, then $\{x_{n_i}\} \in \mathscr{L}$ and $\lim x_{n_i} = x$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

The closure λA of a set $A \subset L$ is defined as the set of all points $\lim x_n$, where $\{x_n\} \in \mathscr{L}$ and $x_n \in A$. In such a way we get a sequential topology or simply $\mathfrak{L}$-topology λ satisfying the following properties:

$$\lambda\emptyset = \emptyset\,, \quad \lambda L = L, \quad \lambda(A \cup B) = \lambda A \cup \lambda B\,, \quad A \subset B \quad \text{implies} \quad \lambda A \subset \lambda B$$
$$\lambda x = x \quad \text{for each} \quad x \in L.$$

The closure of a subset $A \subset L$ need not be closed. Therefore it is possible to form successive closures

$$\lambda^0 A = A \subset \lambda^1 A = \lambda A \subset \lambda^2 A \subset \ldots \subset \lambda^\xi A \subset \ldots \subset \lambda^{\omega_1} A$$

where ω_1 is the first uncountable ordinal and $\lambda^\xi A = \bigcup_{\eta<\xi} \lambda\lambda^\eta A$. It is easy to prove that $\lambda\lambda^{\omega_1} A = \lambda^{\omega_1} A$, so that the set $\lambda^{\omega_1} A$ is the smallest closed set containing A as a subset. Consequently there is no sense in forming a closure $\lambda^\xi A$ for $\xi > \omega_1$.

The usual way of defining the continuity of real functions on L is as follows: f is continuous on L if $f(\lambda A) \subset \overline{f(A)}$ for each subset $A \subset L$. From this definition it follows that a real-valued function f is continuous on L, if and only if $\lim x_n = x$ implies $\lim f(x_n) = f(x)$ for each point $x \in L$. Therefore the continuity of real functions may be called the sequential continuity.

A subset G of an $\mathfrak{L}$-space L is a neighbourhood of a point $x \in L$, if x does not belong to $\lambda(L - G)$. Thus it is possible to define separated convergence spaces S in which any two distinct points are separated by neighbourhoods, and regular convergence spaces R, defined by means of neighbourhood closures. The notion of completely regular convergence space is not suitable for such convergence spaces in which the axiom of the closed closure ($\lambda^2 A = \lambda A$) does not hold true. For convergence spaces we define the notion of sequential regularity (abbr. S-regularity) like this:

The convergence space L is S-regular, if for each point $x_0 \in L$ and each sequence of points $x_n \in L$ no subsequence of which converges to x_0 there is a real valued sequentially continuous function f on L into $\langle 0, 1\rangle$ such that the sequence of numbers $f(x_n)$ fails to converge to $f(x_0)$.

From this definition it follows that the S-regularity of a convergence space is a topological property. In 1947 I constructed a regular $\mathfrak{L}$-space Q such that each continuous function on it is constant [1]. Therefore a regular $\mathfrak{L}$-space need not be S-regular. On the other hand under the supposition that $\aleph_1 = 2^{\aleph_0}$ I was able to construct an S-regular convergence space which is not regular. Consequently regularity and S regularity of sequential topologies are not comparable.

It is well known that each completely regular topological space (fulfilling Kuratowski's axioms of topology) can be characterised as a subspace of a Cartesian cube of a certain dimension the topology of which is the usual topology in the topological product space. On the other hand the following theorem holds true:

A convergence space L is S-regular if and only if it is homeomorphic to a subspace of a Cartesian cube of a certain dimension in which the topology is defined by coordinatewise convergence of real numbers.

This cube will be called an L-cube and denoted by (C, κ).

Now it is possible to define for S-regular convergence spaces a similar notion as the Stone-Čech compactification of completely regular topological spaces.

Let (P, π) be a convergence space contained in an S-regular space (R, ϱ) as a subspace. The convergence space R will be called sequential envelope of the space P if the following conditions c_1, c_2, c_3 are satisfied:

c_1: $R = \varrho^{\omega_1}P$.

c_2: Each sequentially continuous function f on P into $\langle 0,1\rangle$ has a continuous extension $\bar{f}$ on R into $\langle 0, 1\rangle$.

c_3: There is no S-regular convergence space S containing R as a proper subspace and fulfilling the properties c_1 and c_2 relative to P and S.

The following theorem holds true:

Let P be a subspace of an S-regular convergence space R. Then R is a sequential envelope of the space P if and only if there is a homeomorphism h on R onto $\kappa^{\omega_1} \varphi(P) \subset C$ such that $h(x) = \varphi(x)$ for each point $x \in P$, φ being a special homeomorphism on P ($\varphi(x) = \{f_\iota(x)\} \in C$, whereby f_ι runs over all sequentially continuous functions on P into $\langle 0, 1\rangle$) into the $\mathfrak{L}$-cube C, the sequential topology of which is κ.

From this theorem the following statements can be deduced:

Let L' and L'' be two sequential envelopes of the same S-regular $\mathfrak{L}$-space L. Then there exists a homeomorphism h on L' onto L'' such that $h(x) = x$ for each $x \in L$.

Every S-regular convergence space P has a sequential envelope which is homeomorphic to $\kappa^{\omega_1} \varphi(P)$.

The definition of the sequential envelope $\sigma(L)$ of an S-regular convergence space $\mathscr{L}$ is similar to the definition of Stone-Čech compactification $\beta(P)$ of a completely regular topological space P. Nevertheless the properties of the sequential envelope $\sigma(L)$ and of the β-envelope $\beta(L)$ of the same completely regular convergence space L can be completely different. For example the isolated space N of all naturals is a completely regular non-compact space. Consequently $\beta(N) \neq N$. However, it is easy to prove that $\sigma(N) = N$, so that $\beta(N) \neq \sigma(N)$.

The theory of sequential envelopes may be applied to the systems of sets, any system like this being an S-regular convergence space, the convergence in which is defined by the well known condition:

$$\lim A_n = A \quad \text{whenever} \quad A = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n .$$

There is a question whether or not there exists an S-regular $\mathfrak{L}$-space L such that $\sigma(L) \neq L$. The answer is positive. I constructed a space which is homeomorphic to a system of sets, the sequential envelope of which is topologically different from the system itself.

There are some problems concerning the sequential envelope. For instance: What is the sequential envelope of the system of all realvalued functions $f(x)$ of real variable x the convergence in which is defined by the convergence at each point. Or: Of what structure is the sequential envelope of a system of sets.

It is worth noting that for the definition of S-regular convergence spaces and for sequential envelopes Urysohn's axiom of convergence (viz. if $\{x_n\}$ does not converge to x then there is a subsequence $\{x_{n_i}\}$ no subsequence of which converges to x) is not required.

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References

[1] *J. Novák*: Regulární prostor, na němž je každá spojitá funkce konstantní. Časopis pro pěst. mat. a fys., *73* (1948), 58–68.