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ON IMBEDDINGS OF POLYHEDRA INTO EUCLIDEAN SPACES

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Let X be a topological space and E an Euclidean space. A continuous mapping $f : X \to E$ is said to be *k-regular*, if for all $(k + 1)$-tuples of distinct points $x_0, x_1, \ldots, x_k$ of X the points $f(x_0), f(x_1), \ldots, f(x_k)$ are vertices of a k-dimensional simplex in E. For example, the map $f : X \to E$ is 1-regular, iff it is $1 - 1$; the map f is 2-regular iff the points $f(x_0), f(x_1), f(x_2)$ do not lie on a single straight line for distinct points $x_0, x_1, x_2 \in X$. This definition of a k-regular map was given by K. Borsuk.

Let us denote by $F^n(X)$ the set of all continuous maps $X \to E^n$, where E^n is an n-dimensional Euclidean space. Denote by $R_k^n(X)$ the set of all k-regular maps $X \to E^n$.

In connection with the definitions just given the following two problems arise:

Problem 1. (Borsuk). What is the smallest integer n such that the set $R_k^n(X)$ is non-void for all compact metric spaces X of dimension $\leqq p$? This smallest integer will be denoted by $n(p, k)$.

Problem 2. What is the smallest integer n such that the set $R_k^n(X)$ is dense in the metric space $F^n(X)$ for all compact metric spaces X of dimension $\leqq p$. This smallest integer will be denoted by $n'(p, k)$.

It is clear that

$$n(p, k) \leqq n'(p, k) \text{ for all } p \text{ and } k .$$

The Nöbeling-Pontrjagin imbedding theorem states that $n'(p, 1) \leqq 2p + 1$. Furthermore, the example of van Kampen (which is a p-dimensional skeleton of a $(2p + 2)$-dimensional simplex) shows that $n(p, 1) \geqq 2p + 1$. Consequently we have the inequalities

$$2p + 1 \leqq n(p, 1) \leqq n'(p, 1) \leqq 2p + 1$$

and it follows that $n(p, 1) = n'(p, 1) = 2p + 1$. Thus, if $k = 1$ Problem 1 coincides with Problem 2.

It is easy to prove that $n(p, 2) = 2p + 2$. To show this, let X be a p-dimensional compact metric space. Then by the Pontrjagin-Nöbeling theorem, there exists an imbedding map $X \to S^{2p+1}$, where S^{2p+1} is the unit sphere in E^{2p+2}. The composition map $X \to S^{2p+1} \to E^{2p+2}$ is obviously 2-regular (since no three points of a unit sphere can lie on a straight line). Thus we have the equalities $n(p, 1) = 2p + 1$ and $n(p, 2) =$

$= 2p + 2$. These equalities led Borsuk to the conjecture that $n(p, k) = 2p + k$. We shall show immediately that this conjecture is wrong.

We will now state four theorems.

Theorem 1. *The number $n'(p, k)$ is equal to $pk + p + k$.*

The proof is rather complicated; it is given in [1] and uses a generalisation of the notion of intersection number. This generalisation is interesting in its own right.

In virtue of the inequality $n(p, k) \leqq n'(p, k)$, theorem 1 gives us an upper estimate for the number $n(p, k)$:

$$n(p, k) \leqq pk + p + k .$$

We will next consider estimates from below for the number $n(p, k)$. In [2], the two following theorems are proved:

Theorem 2. *Let $f : X \to E^q$ be a k-regular map of the p-dimensional polyhedron X into E^q; then*

$$q \geqq \left[\frac{k+1}{2}\right] p + \left[\frac{k}{2}\right].$$

This theorem holds for an arbitrary polyhedron X. But if we take a sufficiently complicated polyhedron (namely, if X is the p-dimensional skeleton of a cube or simplex of large dimension), then we obtain a stronger estimate as given in the following theorem.

Theorem 3. *We have:*

$$n(p, k) \geqq \begin{cases} pk + p + q\left[\frac{k}{4}\right] & \text{if } k \text{ is odd} , \\ pk + q\left[\frac{k}{4}\right] & \text{if } k \text{ is even} . \end{cases}$$

In particular we have $n(p, 3) \geqq 3p$, which shows that Borsuk's conjecture is wrong. More precisely, Borsuk conjectured that the number k appears in $n(p, k)$ as a summand, but in fact k appears in the estimate of $n(p, k)$ from below as a multiplicative factor: $n(p, k) \geqq pk$.

In [2], we obtained an interesting application of the above theorems to the constructive theory of functions. In order to formulate this application we shall introduce some definitions.

Let X be a compact metric space and $C(X)$ the space of all real-valued continuous functions on X with the usual norm

$$\|f\| = \max_{x \in X} |f(x)| .$$

Furthermore, suppose that

$$f_0(x) \equiv 1 , \quad f_1(x), \ldots, f_m(x)$$

are linearly independent elements in the Banach space $C(X)$. Let us denote by L_m the linear subspace of $C(X)$ generated by the elements $f_0, f_1, \ldots, f_m$. In the constructive theory of functions, the following problem of Chebyshev plays an important rôle:

For a given function $\varphi \in C(X)$, find the polynomial of best approximation for the system $(f_0, f_1, \ldots, f_m)$; that is, find an element $p^* \in L_m$ such that

$$\|\varphi - p^*\| = \min_{p \in L_m} \|\varphi - p\|.$$

A solution of Chebyshev's problem always exists, but in general it is not unique. The set $V(\varphi)$ of all polynomials giving the best approximation is a convex set in L_m, which is called the polyhedron of best approximation. The number

$$\max_{\varphi \in C(X)} \dim V(\varphi)$$

is called the Chebyshev rank of the system $(f_0, f_1, \ldots, f_m)$.

Theorem 4. *Let X be a p-dimensional polyhedron and m a positive integer. Then the Chebyshev rank of any system $(f_0 \equiv 1, f_1, \ldots, f_m)$ is not less than* $\frac{p-1}{p+1} m - \frac{q}{p+1}$. *Furthermore, there exists a system $f_0 \equiv 1, f_1, \ldots, f_m$ on X such that its Chebyshev rank is not more than* $\frac{p}{p+1} m + \frac{2p+1}{p+1}$.

In particular, if $\dim X \geqq q$, then for $m \to \infty$ the Chebyshev rank of systems $(f_0 \equiv 1, f_1, \ldots, f_m)$ increases at least as quickly the linear function $\lambda m + \mu$, where $\lambda = \frac{p-1}{p+1} > 0$; thus the Chebyshev rank tends to infiinity. In other words, only on one-dimensional polyhedra can there be a system $(f_0 \equiv 1, f_1, \ldots, f_m)$ of bounded Chebyshev rank and arbitrary length m.

References

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