ON SOME SPACES OF FUNCTIONS AND DISTRIBUTIONS

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In [4] L. SCHWARTZ introduced spaces $\mathscr{D}_{L^p}$ of functions and $\mathscr{D}'_{L^p}$ of distributions. The purpose of this note is to present some properties of spaces $\mathscr{D}_M$ and $\mathscr{D}'_M$ replacing spaces $\mathscr{L}^p$ in Schwartz's definition by Orlicz spaces $\mathscr{L}^*_M$.[1]) Let $M(u)$ be an even, continuous, convex, nonnegative function assuming the value 0 only at $u = 0$, $u^{-1} M(u) \to 0$ as $u \to 0$ and $u^{-1} M(u) \to \infty$ as $u \to \infty$. We define

$$\mathscr{D}_M = \bigcap_p \{\varphi \in \mathscr{E} : \int M(k_p D^p \varphi(x))\, \mathrm{d}x < \infty, \text{ where } k_p > 0 \text{ depends on } \varphi\} ;$$

here $\mathscr{E}$ is the space of all infinitely differentiable functions of n variables, the integral is taken over the whole n-dimensional space and the product $\bigcap_p$ runs over all systems $p = (p_1, \ldots, p_n)$ of nonnegative integers. Defining the topology in $\mathscr{D}_M$ by a countable system of seminorms

$$\|D^p\varphi\|_M = \inf \{\varepsilon > 0 : \int M(\varepsilon^{-1} D^p \varphi(x))\, \mathrm{d}x \leqq 1\} ,$$

$\mathscr{D}_M$ becomes a B_0-space. We denote by $\mathscr{D}'_N$ the dual of $\mathscr{D}_M$, where $N(u)$ is the function complementary to $M(u)$ in the sense of Young.

The following elementary properties hold:

If $\varphi \in \mathscr{D}_M$ then $\varphi(x) \to 0$ as $|x| \to \infty$; if $\varphi_k \to 0$ in $\mathscr{D}_M$ then $\varphi_k(x)$ are uniformly bounded and $\varphi_k(x) \to 0$ as $|x| \to \infty$ uniformly in k. Assuming $M_2(u) = 0(M_1(u))$ as $u \to 0$, we have

$$\mathscr{D}_{M1} \mathrel{\dot{\subset}} \mathscr{D}_{M2} \quad \text{and} \quad \mathscr{D}'_{N_2} \mathrel{\dot{\subset}} \mathscr{D}'_{N_1} ;$$

here $\mathscr{X} \mathrel{\dot{\subset}} \mathscr{Y}$ means that $\mathscr{X}$ is a part of $\mathscr{Y}$ with a finer topology. Moreover, we have $\mathscr{L}^*_N \mathrel{\dot{\subset}} \mathscr{D}'_N$. If $M(u)$ and $N(u)$ satisfy the condition (Δ_2): $M(2u) \leqq \kappa\, M(u)$ with a $\kappa > 0$ for all u, then the set $\mathscr{D}$ of all infinitely differentiable functions of compact support is dense in $\mathscr{D}_M$ and in $\mathscr{D}'_N$, whence $\mathscr{D}'_N$ is a normal space of distributions, the space $\mathscr{D}_M$ is reflexive and $\mathscr{D}'_N$ consists exactly of finite sums of (distributional) derivatives of functions belonging to $\mathscr{L}^*_N$.

In the above introduced spaces, the integral transform

$$K\, \varphi(x) = \int k(x, y)\, \varphi(y)\, \mathrm{d}y$$

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[1]) For the proofs of results presented here, cf. [1], [2] [3].

and its adjoint K^* defined by $K^* T(\varphi) = T(K\varphi)$, where T is a distribution, may be considered. Assume M_1, M_2, N_1, N_2 satisfy the condition (Δ_2) for all u, and x and y are points of the n-dimensional and m-dimensional space, respectively. Let $k(x, y)$ be an infinitely differentiable function of x for every y, $k(x, y)$ measurable in the $(n + m)$-dimensional space. Finally, let $k(x, y)$ satisfy the following assumptions (As):

1° $D_x^p k(x, y)$ is a function of x equicontinuous in every bounded set of y,

2° $k_p(x) = \|D_x^p k(x, .)\|_{M_2}$ is bounded for every p separately,

3° $\|k_p\|_{N_1}$ is finite for every p.

Then K and K^* are linear compact operators from $\mathscr{L}^*_{N_2}$ to $\mathscr{D}_{N_1}$ and from $\mathscr{D}'_{M_1}$ to $\mathscr{L}^*_{M_2}$, respectively, and the ranges of K and K^* are linear subspaces of the first category in $\mathscr{D}_{N_1}$ resp. $\mathscr{L}^*_{M_2}$. If, moreover, $\|D_x^p k(., y)\|_{N_1}$ is bounded in y for every p separately and the support of $k(x, y)$ is contained in a strip $\{(x, y) : y \in A\}$, where A is of finite measure in the m-dimensional space, then

$$K^* T(y) = \int k(x, y)\, T_x \, \mathrm{d}x$$

for every $T \in \mathscr{D}'_{M_1}$, the last integral being defined in Schwartz's sense [5].

Besides spaces $\mathscr{D}'_N$, spaces $\overline{\mathscr{D}}'_N(E)$ of vector-valued distributions (cf. e. g. [5]) may be considered, where $\mathscr{H}(E) = \mathscr{L}_\varepsilon(\mathscr{H}'; E)$ is the space of linear continuous operations from $\mathscr{H}'$ to E provided with the topology of uniform convergence on equicontinuous parts of $\mathscr{H}'$ (here E and $\mathscr{H}$ are locally convex linear topological Hausdorff spaces and $\mathscr{H}$ is a space of distributions). Of course, $\overline{\mathscr{D}}_{N_1}(\mathscr{L}^*_{M_2})$ consists of linear operators adjoint to operators from $\overline{\mathscr{L}}^*_{M_2}(\mathscr{D}_{N_1})$; examples of such operators yield the above considered transforms K and K^*. It is easily seen that taking as E a Banach space and denoting by $\mathscr{L}^*_M[E]$ the space of all vector-valued functions with values in E, M-integrable in Bochner's sense, i. e.

$$\mathscr{L}^*_M[E] = \{f : f(x) \text{ is strongly measurable and } \int M(k\|f(x)\|)\, \mathrm{d}x < \infty$$
$$\text{for a } k > 0 \text{ dependent on } f\}$$

where $f = g$ means that $f(x) = g(x)$ almost everywhere) with the norm

$$\|f\|_M = \inf\{\varepsilon > 0 : \int M(\varepsilon^{-1}\|f(x)\|)\, \mathrm{d}x \leqq 1\},$$

we have

$$\mathscr{L}^*_M[E] \mathrel{\dot{\subset}} \overline{\mathscr{D}}'_M(E).$$

References

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[5] *L. Schwartz*: Séminaire 1953/54. Paris 1954.