DIFFERENTIAL STRUCTURES

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0. In the paper "*Strutture differenziali e varieta' di classe* $\mathscr{C}^1$", published in the Annali della Scuola Normale Superiore di Pisa, vol. XVIII (1964), pp. 343–365, (to be referred to in the sequel as [SD]), the author has given a definition of differential structure on a topological space and, consequently, of differentiable functions on the space so structured having in mind the scheme of inverse topologies deduced from a family of real functions.

Here we deal with the same subject though with a slight change in the initial part of the definition. Such alteration was evolved in the consideration of certain examples and from development of the theory.

Following the presentation of the main definitions we give certain properties of differentiable functions. Then, after the consideration of some significant examples, we take up the differentiability of applications between differentiable varieties. This theory, as yet, is far from being complete.

Proofs of some of the following propositions may be drawn from those appearing in [SD]. The complete proofs, however, are to be found in a paper on this subject which, the author hopes, will appear soon.

1. Notational conventions:

(a) The symbol $\subset$ between sets does not exclude equality.

(b) $\boldsymbol{R}$ will always mean the set of reals endowed with the usual structure of topological field; whenever vector spaces or linear mappings are considered, they are always referred to $\boldsymbol{R}$ as ground field.

(c) Suppose A and B are sets and $f : A \to B$ is a mapping; if $Y \subset A, f(Y)$ will denote the set of images of all elements of Y under f.

(d) If A is a set, $A^{\#}$ will denote the vector space af all mappings of A into $\boldsymbol{R}$. The space $A^{\#}$ will be considered only when A is not empty.

(e) If $f : A \to B$ is a mapping, $f^{\#}$ will denote the so-called *adjoint mapping of* $B^{\#}$ into $A^{\#}$ which to every $g \in B^{\#}$ assigns $g \circ f \in A^{\#}$. The same symbol $f^{\#}$ will be used to denote restrictions to subsets of $B^{\#}$.

(f) If A is a set and $a \in A$, π_a will denote the mapping of $A^{\#}$ (or of one of its subsets) into $\boldsymbol{R}$ such that $\pi_a(g) = g(a)$, for every $g \in A^{\#}$. Thus π is a mapping of A into $A^{\#\#}$ (or into $Y^{\#}$, for some $Y \subset A^{\#}$).

(g) If A is a topological space, $\mathscr{C}^\circ(A)$ is the vector space of all continuous functions of A into $\boldsymbol{R}$ (a slightly different use of this symbol is made in no. 2.).

2. We recall first some well-known topological facts expressed so as to show the analogy with what follows.

Let A be a set and $\mathscr{G}$ a set of mappings of A into $\boldsymbol{R}$. If we consider on the set A the topology generated by the set $\mathscr{G}$ (i.e., the weakest topology under which all elements of $\mathscr{G}$ are continuous), we get a completely regular space, the set of whose continous real functions is denoted by $\mathscr{C}^\circ(A, \mathscr{G})$, or, simply, by $\mathscr{C}^\circ(\mathscr{G})$ when no confusion can arise. The set $\mathscr{C}^\circ(\mathscr{G})$ comes out to be a vector space containing the family $\mathscr{G}$ and all constant real functions on A.

We define now for every $\mathscr{H}, \mathscr{G} \subset A^\#$, $\mathscr{H} <^\circ \mathscr{G}$ to mean that $\mathscr{C}^\circ(\mathscr{H}) \subset \mathscr{C}^\circ(\mathscr{G})$ and, further, $\mathscr{H} \sim^\circ \mathscr{G}$ to mean that $\mathscr{C}^\circ(\mathscr{H}) = \mathscr{C}^\circ(\mathscr{G})$. Then the following facts hold:

I. *If* $\mathscr{H}, \mathscr{G} \subset A^\#$, *then*

(a) $\mathscr{G} \subset \mathscr{H} \Rightarrow \mathscr{G} <^\circ \mathscr{H}$;
(b) $\mathscr{G} \subset \mathscr{H} \subset \mathscr{C}^\circ(\mathscr{G}) \Rightarrow \mathscr{G} \sim^\circ \mathscr{H}$;
(c) *if* $|\mathscr{G}|^\circ$ *is the class of all families* $\mathscr{H} \subset A^\#$ *such that* $\mathscr{G} \sim^\circ \mathscr{H}$, *then* $\mathscr{C}^\circ(\mathscr{G}) = \bigcup\{\mathscr{H} : \mathscr{H} \in |\mathscr{G}|^\circ\}$.

The topological space obtained from the set A starting with the family $\mathscr{G}$ is called a $\mathscr{C}^\circ$-variety and is denoted by $(A, \mathscr{G})$. It is obvious that two varieties $(A, \mathscr{G})$ $(A, \mathscr{H})$ have the same topological structure if and only if $\mathscr{G} \sim^\circ \mathscr{H}$: this means that in fact the variety $(A, \mathscr{G})$ should be written as $(A, |\mathscr{G}|^\circ)$.

If $(A, \mathscr{G})$, $(B, \mathscr{H})$ are two $\mathscr{C}^\circ$-varieties and $g : A \to B$, we say that the mapping g is continuous to mean that $g^\#(\mathscr{H}) <^\circ \mathscr{G}$.

3. Let now A be a topological space and $\mathscr{G}$ be a family of real functions on A. Let $[\mathscr{G}]$ be the vector space spanned by $\mathscr{G}$ in $A^\#$ and let $[\mathscr{G}]^*$ be the algebraic dual of $[\mathscr{G}]$. With $\mathscr{G}^*$ we denote the set of the restrictions of all elements of $[\mathscr{G}]^*$ to $\mathscr{G}$. It is easily seen that $\mathscr{G}^*$ is again a vector space and the mapping $i^\# : [\mathscr{G}]^* \to \mathscr{G}^*$ (adjoint of the inclusion of $\mathscr{G}$ into $[\mathscr{G}]$) is an isomorphism.

On the space $[\mathscr{G}]^*$ we consider the $\sigma([\mathscr{G}]^*, [\mathscr{G}])$-topology (a kind of w^*-topology). It is the weakest topology under which the images π_f of all $f \in [\mathscr{G}]$, in the algebraic bidual of $[\mathscr{G}]$, are continuous. We thereby obtain a locally convex topological vector space. Notice that the same topology is obtained if $\mathscr{G}$ instead of $[\mathscr{G}]$ were considered (i.e., it coincides with the $\sigma([\mathscr{G}]^*, \mathscr{G})$-topology).

Finaily, we consider on $\mathscr{G}^*$ the topology induced by the mapping $i^\#$ (i.e., the finest topology under which $i^\#$ is continuous). This topology may be viewed as the $\sigma(\mathscr{G}^*, (i^{\#\#})^{-1}(\pi(\mathscr{G})))$-topology (i.e., the weakest topology under which, for every $f \in \mathscr{G}$, if $\pi_f = \pi_g \circ i^\#$ (such a function π_g always exists), π_g is continuous). We note that $\mathscr{G}^*$ too is a locally convex topological vector space.

In the product space $A \times A \times \mathscr{G}^*$ let us consider the set

$$\mathscr{I}(A, \mathscr{G}) = \{(a, b, \pi_a - \pi_b) : \text{for all } a, b \in A\}$$

and let us denote by $\mathscr{D}(A, \mathscr{G})$ the least (with respect to inclusion) closed set in $A \times \times A \times \mathscr{G}^*$ which contains $\mathscr{I}(A, \mathscr{G})$ and is such that for every $a, b \in A$

$$V_{a,b} = \{\lambda : (a, b, \lambda) \in \mathscr{D}(A, \mathscr{G})\}$$

is a subspace of the vector space $\mathscr{G}^*$. For every $a \in A$, the set

$$T_a(A, \mathscr{G}) \quad (\text{or, simply, } T_a) = \{(a, \lambda) : (a, a, \lambda) \in \mathscr{D}(A, \mathscr{G})\}$$

is a topological vector space (with the structure induced by the natural bijection with the space $V_{a,a}$) which is called the *tangent space at a*. The couples $(a, \lambda) \in T_a$ are called *tangent vectors at a*. Finally, the *tangent bundle*, as usual, is defined to be the set

$$T(A, \mathscr{G}) = \{(a, \lambda) : a \in A,\ (a, \lambda) \in T_a(A, \mathscr{G})\}\ .$$

4. Let now $f : A \to \boldsymbol{R}$ be a function, let σ_f be the function from $\mathscr{I}(A, \mathscr{G})$ to $\boldsymbol{R}$ which takes (a, b, λ) into $\lambda(f)$. Notice that in the subspace topology ($\mathscr{I}(A, \mathscr{G}) \subset \subset A \times A \times \mathscr{G}^*$) the function σ_f is continuous and, moreover, it is linear in the third argument.

The function f is said to be *differentiable* if and only if the function σ_f can be extended to the whole of $\mathscr{D}(A, \mathscr{G})$ so as to remain continuous and linear in the third argument. Such an extension, denoted by $\tilde{\sigma}_f$, when it exists is unique and linear with respect to f: meaning that if, for f and g in $A^\#$, $\tilde{\sigma}_f$ and $\tilde{\sigma}_g$ exist, then, for $k \in \boldsymbol{R}$, also $\tilde{\sigma}_{kf+g}$ exists and is equal to $k\tilde{\sigma}_f + \tilde{\sigma}_g$.

The family of all differentiable functions is denoted by $\mathscr{C}^1(A, \mathscr{G})$. It is a vector space and contains all constant real functions on A.

For every differentiable function f we define the *differential of f* to be the function $df : T(A, \mathscr{G}) \to \boldsymbol{R}$ which to every (a, λ) associates $\tilde{\sigma}_f(a, a, \lambda)$. The *differential of f at a*, denoted by df_a, is the restriction of the differential df to the tangent space $T_a(A, \mathscr{G})$. Finally, as *differential morphism on* $(A, \mathscr{G})$ we define the mapping $d : \mathscr{C}^1(A, \mathscr{G}) \to (T(A, \mathscr{G}))^\#$ which takes every differentiable function into its differential.

As for the continuity, if $\mathscr{G}, \mathscr{H} \subset A^\#$, we define $\mathscr{G} <^1 \mathscr{H}$ to mean that $\mathscr{C}^1(A, \mathscr{G}) \subset \subset \mathscr{C}^1(A, \mathscr{H})$ and $\mathscr{G} \sim^1 \mathscr{H}$ to mean that $\mathscr{C}^1(A, \mathscr{G}) = \mathscr{C}^1(A, \mathscr{H})$.

It is useful to notice that:

II. *For every* $\mathscr{G}, \mathscr{H} \subset A^\#$,

(a) $\mathscr{G} \subset \mathscr{C}^1(A, \mathscr{G})$;

(b) $\mathscr{G} \subset \mathscr{H} \Rightarrow \mathscr{G} <^1 \mathscr{H}$.

We want now to make precise the feeling that differentiable functions and tangent vectors determine each other; in the sense that if the tangent bundles coming from two families of functions are in a certain sense 'isomorphic' then the two families are $\sim^1$-equivalent, and conversely. To this end we present a definition and follow it by a proposition which exhibits some useful properties of the defined objects.

Let $\mathscr{G}$, $\mathscr{H}$ be parts of $A^{\#}$ and let $\mathscr{G}$ be contained in $\mathscr{H}$. Then the identity mapping i on A is such that $i^{\#}(\mathscr{G}) \subset \mathscr{H}$ and $i^{\#\#}(\mathscr{H}^*) \subset \mathscr{G}^*$. Thus we are able to define the mapping $\eta : A \times A \times \mathscr{H}^* \to A \times A \times \mathscr{G}^*$ in such a way that $\eta(a, b, \lambda) = (a, b, i^{\#\#}(\lambda)) = (a, b, \lambda/\mathscr{G})$. This mapping, called the *canonical extension* of the identity mapping, turns out to be continuous and linear in the third argument. Let now $\mathscr{G}$ and $\mathscr{H}$ be two arbitrary subsets of $A^{\#}$. We define $\mathscr{D}(A, \mathscr{G}) \approx \mathscr{D}(A, \mathscr{H})$ to mean that the canonical extensions of the identity on A

$$\eta_1 : A \times A \times (\mathscr{G} \cup \mathscr{H})^* \to A \times A \times \mathscr{G}^*$$

and

$$\eta_2 : A \times A \times (\mathscr{G} \cup \mathscr{H})^* \to A \times A \times \mathscr{H}^* ,$$

restricted to the set $\mathscr{D}(A, \mathscr{G} \cup \mathscr{H})$ are both homeomorphisms, respectively, onto the sets $\mathscr{D}(A, \mathscr{G})$ and $\mathscr{D}(A, \mathscr{H})$ (consequently, for every $a \in A$, they induce bicontinuous isomorphisms between $T_a(A, \mathscr{G} \cup \mathscr{H})$, $T_a(A, \mathscr{G})$ and $T_a(A, \mathscr{H})$).

Analogously, we define $T(A, \mathscr{G}) \approx T(A, \mathscr{H})$ to mean that the restrictions of the mappings η_1 and η_2 to the set $T(A, \mathscr{G} \cup \mathscr{H})$ are homeomorphisms, respectively, onto the sets $T(A, \mathscr{G})$ and $T(A, \mathscr{H})$.

III. *If $\mathscr{G}$ and $\mathscr{H}$ are subsets of $A^{\#}$, the following propositions are equivalent:*

(a) $\mathscr{G} <^1 \mathscr{H}$;
(b) $\mathscr{G} \subset \mathscr{C}^1(A, \mathscr{H})$;
(c) $\mathscr{D}(A, \mathscr{H}) \approx \mathscr{D}(A, \mathscr{G} \cup \mathscr{H})$;
(d) $\mathscr{G} \cup \mathscr{H} <^1 \mathscr{H}$, *hence* $\mathscr{G} \cup \mathscr{H} \sim^1 \mathscr{H}$;
(e) $\mathscr{C}^1(A, \mathscr{G} \cup \mathscr{H}) = \mathscr{C}^1(A, \mathscr{H})$.

From this proposition we get the corollary:

IV. *If $\mathscr{G}, \mathscr{H} \subset A^{\#}$, then the following propositions are equivalent:*

(a) $\mathscr{G} \sim^1 \mathscr{H}$;
(b) $\mathscr{D}(A, \mathscr{G}) \approx \mathscr{D}(A, \mathscr{H})$;
(c) $T(A, \mathscr{G}) \approx T(A, \mathscr{H})$.

Other properties of differentiable functions are expressed in the following proposition:

V. *If $\mathscr{G}, \mathscr{H} \subset A^{\#}$, then:*

(a) $\mathscr{G} \sim^1 \mathscr{C}^1(A, \mathscr{G})$;
(b) $f \in \mathscr{C}^1(A, \mathscr{G}) \Leftrightarrow \{f\} \cup \mathscr{G} \sim^1 \mathscr{G}$;
(c) $\mathscr{C}^1(A, \mathscr{G}) = \bigcup\{\mathscr{H} : \mathscr{H} \subset A^\# \text{ and } \mathscr{H} \sim^1 \mathscr{G}\}$;
(d) $\mathscr{G} \subset \mathscr{H} \subset \mathscr{C}^1(A, \mathscr{G}) \Rightarrow \mathscr{C}^1(A, \mathscr{G}) = \mathscr{C}^1(A, \mathscr{H})$.

With regard to the continuity of differentiable functions we have the following:

VI. *If $\mathscr{G}, \mathscr{H} \subset A^\#$, then $\mathscr{H} <^1 \mathscr{G} \subset \mathscr{C}^0(A) \Rightarrow \mathscr{H} \subset \mathscr{C}^0(A)$. Hence, if the family $\mathscr{G}$ contains only continuous functions then every differentiable function is continuous.*

Finally, we notice that if $|\mathscr{G}|^1$ denotes the class of all families $\mathscr{H} \subset A^\#$ such that $\mathscr{H} \sim^1 \mathscr{G}$, then $|\mathscr{G}|^1$, more properly than $\mathscr{G}$, determines the differential structure on the space A. For this reason we say that $|\mathscr{G}|^1$ defines a *differential structure on A* and we call the pair $(A, |\mathscr{G}|^1)$ a *differentiable manifold* or, better, a *$\mathscr{C}^1$-variety*. However, in practice we will write simply $(A, \mathscr{G})$; of course it should be understood that $(A, \mathscr{G}) = (A, \mathscr{H})$ if and only if $\mathscr{G} \sim^1 \mathscr{H}$ (i.e., $|\mathscr{G}|^1 = |\mathscr{H}|^1$).

5. Some examples of differentiable manifolds will now be exhibited.

(a) In [SD, prop. XX] we have shown that in the $\mathscr{C}^1$-variety $(\boldsymbol{R}^n, \mathscr{G})$, where $\boldsymbol{R}^n$ is the usual n-dimensional space and $\mathscr{G}$ is the family of the n projections $\pi_1, \dots, \pi_n$ from $\boldsymbol{R}^n$ onto $\boldsymbol{R}$, it is true that:

(a.1) $\mathscr{C}^1(\boldsymbol{R}^n, \mathscr{G})$ is the class of all functions from $\boldsymbol{R}^n$ to $\boldsymbol{R}$ which are continuous with their first derivatives;

(a.2) for every $a \in \boldsymbol{R}^n$, the tangent space T_a is isomorphic to $\boldsymbol{R}^n$;

(a.3) for every $f \in \mathscr{C}^1(\boldsymbol{R}^n, \mathscr{G})$ and for every $a \in \boldsymbol{R}^n$,

$$df_a = (\mathrm{D}_1 f)_a \, d\pi_1 + \dots + (\mathrm{D}_n f)_a \, d\pi_n ,$$

where $(\mathrm{D}_j f)_a$ denotes the j-th derivative of f at the point a.

(b) Let $\beta : \boldsymbol{R}^n \to \boldsymbol{R}^p$ (with $n < p + 1$) be a continuous mapping with continuous first derivatives. If $A = \beta(\boldsymbol{R}^n)$ and $\mathscr{G} = \{\pi_1, \dots, \pi_p\}$, then $\mathscr{G}^*$ is $\boldsymbol{R}^p$ and for every $a \in A$, if $a = \beta(x)$ and the jacobian of β does not vanish at x, then the tangent space to $(A, \mathscr{G})$ at a has dimension not less than n. If, moreover, $x = \beta^{-1}(a)$, then the dimension of T_a is n.

(c) We notice that the tangent space T_a at the point a of a variety, which is embedded in a $\boldsymbol{R}^n$ space, contains the (intuitively understood) tangent cone at a. However, the dimension of T_a is not always the least which is compatible with this property: e.g., if $\beta : \boldsymbol{R} \to \boldsymbol{R}^2$ is such that $\beta(x) = (x^2, x^3)$ for every $x \in \boldsymbol{R}$, then the tangent cone at $(0, 0)$ contains only the subspace $\{(x, 0) : x \in \boldsymbol{R}\}$, whereas the tangent space $T_{(0,0)}$ is $\boldsymbol{R}^2$.

(d) Let A be a locally convex topological vector space. Let $\mathscr{G}$ be the topological dual A' of A. Then $\mathscr{G}^*$ is the algebraic dual of A' with the w^*-topology. In the space

$A \times A \times \mathscr{G}^*$ the set $\mathscr{D}(A, \mathscr{G})$ is constituted of all elements $(a, b, k(\pi_a - \pi_b))$ where $a, b \in A$ and $k \in \boldsymbol{R}$, together with all triples (a, a, λ) with $a \in A$ and $\lambda \in \mathscr{G}^*$, since the set $\pi(A)$ is dense in $\mathscr{G}^*$.

It follows that every differentiable function f on $(A, \mathscr{G})$ is strongly differentiable in the following sense: for every $a, b \in A$, $b \neq 0$, the function from $\boldsymbol{R}$ to $\boldsymbol{R}$ which to every $k \in \boldsymbol{R}$ associates $f(a + kb)$ is derivable at $k = 0$, and the derivative $\delta f(a, b)$ (sometimes called the GÂTEAUX-variation of f) is continuous with respect to both arguments on the product $A \times A$, where the first factor has the initial topology while the second one has the weak topology inherited from the embedding into $\mathscr{G}^*$. It follows that $\delta f(a, b)$ is continuous also in the product $A \times A$ when both factors have the initial topology. So, if A is a normed space, every differentiable function on $(A, \mathscr{G})$ is FRÉCHET-differentiable with continuous derivative.

6. Let now $(A, \mathscr{G})$ and $(B, \mathscr{H})$ be two $\mathscr{C}^1$-varieties. We say that the mapping $f : A \to B$ is $(\mathscr{G}, \mathscr{H})$-*differentiable* (or simply *differentiable*, when no confusion is likely to arise) if and only if $f^{\#}(\mathscr{C}^1(B, \mathscr{H})) <^1 \mathscr{G}$. If f is a differentiable mapping, its *differential* df is defined to be the mapping from $T(A, \mathscr{G})$ to $T(B, \mathscr{H})$ which takes every (a, λ) into $(f(a), \mu)$, where $\mu \in \mathscr{H}^*$ is such that for every $h \in \mathscr{H}$, $\mu(h) = \tilde{\sigma}_{f^{\#}(h)}(a, a, \lambda)$. As usual, the *differential of f at a* is the restriction df_a of df to the tangent space $T_a(A, \mathscr{G})$.

We notice that the differential is well defined since both $\tilde{\sigma}_y$ and $f^{\#}(y)$ are linear with respect to y.

Noteworthy properties of differential mappings are expressed in the following propositions:

VII. *The differentiability and the differential of a mapping $f : A \to B$ depend on the differential structures determined by the classes $|\mathscr{G}|^1$ and $|\mathscr{H}|^1$ and not on the actual families $\mathscr{G}$ and $\mathscr{H}$. That is, for $\mathscr{G}, \mathscr{L} \subset A^{\#}$ and $\mathscr{H}, \mathscr{M} \subset B^{\#}$, if $\mathscr{G} \sim^1 \mathscr{L}$ and $\mathscr{H} \sim^1 \mathscr{M}$, then*

$$f \text{ is } (\mathscr{G}, \mathscr{H})\text{-differentiable} \Leftrightarrow f \text{ is } (\mathscr{L}, \mathscr{M})\text{-differentiable}.$$

VIII. *If the mapping $f : A \to B$ is continuous, then*

$$f \text{ is } (\mathscr{G}, \mathscr{H})\text{-differentiable} \Leftrightarrow f^{\#}(\mathscr{H}) <^1 \mathscr{G}.$$

IX. *Let $\mathscr{G}$ be a family of continuous functions on A to $\boldsymbol{R}$. If the continuous functions on the topological space B are determined by the family $\mathscr{H}$ in the sense that there exists $\mathscr{L} \subset B^{\#}$ such that $\mathscr{H} \sim^1 \mathscr{L}$ and $\mathscr{L} <^\circ \mathscr{C}^\circ(B)$, then every $(\mathscr{G}, \mathscr{H})$-differentiable mapping is also continuous. Moreover to ensure $(\mathscr{G}, \mathscr{H})$-differentiability of f it is sufficient that $f^{\#}(\mathscr{L}) <^1 \mathscr{G}$.*

X. *If $\mathscr{G}, \mathscr{H} \subset A^{\#}$, then $\mathscr{H} <^1 \mathscr{G}$ is equivalent to the $(\mathscr{G}, \mathscr{H})$-differentiability of the identity mapping on A.*

XI. *Differentiability is preserved under composition. If* $(A, \mathscr{G})$, $(B, \mathscr{H})$, $(E, \mathscr{L})$ *are* $\mathscr{C}^1$*-varieties and the mappings* $f : A \to B$, $g : B \to E$ *are, respectively,* $(\mathscr{G}, \mathscr{H})$*- and* $(\mathscr{H}, \mathscr{L})$*-differentiable, then composite mapping* $g \circ f$ *is* $(\mathscr{G}, \mathscr{L})$*-differentiable.*

XII. *The differential df of a (differentiable) function f is continuous in both arguments. The differential* df_a *at the point a, is a continuous linear mapping from the tangent space at a into the tangent space at* $f(a)$.

XIII. *Suppose* $(A, \mathscr{G})$ *and* $(B, \mathscr{H})$ *are* $\mathscr{C}^1$*-varieties. Let* d_A *and* d_B *denote, respectively, their differential morphisms. If* $g : A \to B$ *is a differentiable mapping, then* $d_A \circ g^{\#} = (dg)^{\#} \circ d_B$. *That is, the following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{C}^1(A, \mathscr{G}) & \xrightarrow[d_A]{} & (T(A, \mathscr{G}))^{\#} \\ {\scriptstyle g^{\#}}\uparrow & & \uparrow{\scriptstyle (dg)^{\#}} \\ \mathscr{C}^1(B, \mathscr{H}) & \xrightarrow[d_B]{} & (T(B, \mathscr{H}))^{\#} \end{array}$$

XIV. *Let i be the identity mapping on* $\boldsymbol{R}$, *then, for every* $\mathscr{G} \subset A^{\#}$, *the* $(\mathscr{G}, \{i\})$*-differentiability for a given function* $g : A \to \boldsymbol{R}$ *coincides with the differentiability of g itself as defined in no. 4. Hence proposition* VI *becomes a special case of proposition* IX.

This last proposition may be completed noticing that also the differential of the function g as given in this no. coincides with the differential dg as defined in no. 4. We have only to bear in mind that actually the differential of a real function g at a point a, corresponding to the vector h, is the vector $g'(a)\, h$ attached to the point $g(a)$, which is usually omitted.

Considering $\boldsymbol{R}^n$ with the usual differential structure and noticing that a mapping $f : A \to \boldsymbol{R}^n$, $f = (f_1, \ldots, f_n)$, is differentiable if and only if so are the functions $f_j : A \to \boldsymbol{R}$, we have that:

XV. *If* $(A, \mathscr{G})$ *is a* $\mathscr{C}^1$*-variety, the class* $\mathscr{C}^1(A, \mathscr{G})$ *of differentiable functions is differentially complete in the sense that if* $g_1, \ldots, g_n$ *are differentiable functions of A into* $\boldsymbol{R}$, *and f is a differentiable function from* $\boldsymbol{R}^n$ *into* $\boldsymbol{R}$, *then the function* $f \circ (g_1, \ldots, g_n) : A \to \boldsymbol{R}$ *is differentiable.*

Finally we observe that the differentiability has a local character in the sense that if we define a function f from the $\mathscr{C}^1$-variety $(A, \mathscr{G})$ into $\boldsymbol{R}$ to be *differentiable near the point* $a \in A$ if and only if there exists an open set $T \subset A$ to which a belongs and such that $f/_T = i^{\#}(f) \in i^{\#}(\mathscr{C}^1(A, \mathscr{G}))$, $i^{\#}$ being the adjoint mapping of the inclusion of the subset T into A, then we have:

XVI. *For any given* $\mathscr{C}^1$*-variety* $(A, \mathscr{G})$, *the function* $f : A \to \boldsymbol{R}$ *is differentiable if and only if it is differentiable near every point of A.*

7. We conclude with an open question which, we believe, is of some interest.

Let $(A, \mathscr{G})$ be a $\mathscr{C}^1$-variety and let B be a topological space. If a mapping $h : B \to A$ is given, we may ask under which conditions there exists a family of functions $\mathscr{L} \subset B^\#$ such that $h^\#(\mathscr{C}^1(A, \mathscr{G})) = \mathscr{C}^1(B, \mathscr{L})$.

We notice that if the mapping h is continuous and if such a family $\mathscr{L}$ does exist, then, since $\mathscr{L} \sim^1 h^\#(\mathscr{G})$, the following equality holds $h^\#(\mathscr{C}^1(A, \mathscr{G})) = \mathscr{C}^1(B, h^\#(\mathscr{G}))$.

In particular, if $B \subset A$ and if h is the inclusion mapping, the last equality gives a WHITNEY-type theorem on extension of differentiable mappings.

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