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# PRECLOSED MULTIVALUED MAPPINGS

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The definition of preclosed univalued mappings has been given in our paper [4]. The purpose of the recent note is to give the definition of preclosed multivalued mappings and to show some results concerning these mappings.

Let $f : X \to Y$ denote a multivalued mapping from a topological space $X$ onto a topological space $Y$, and let $A$ be a subset of $X$, $B$ a subset of $Y$.

Following V. I. Ponomarev [1] the set

$$f^{-1}B = \mathrm{E}\{x \mid x \in X,\ fx \cap B \neq \emptyset\}$$

will be called the *large inverse image* of $B$, the set

$$f_b^{-1}B = \mathrm{E}\{x \mid x \in X,\ fx \subseteq B\}$$

– the *small inverse image* of $B$, the set

$$fA = \mathrm{E}\{y \mid y \in Y,\ f^{-1}y \cap A \neq \emptyset\}$$

– the *large image* of $A$, and finally, the set

$$f_bA = \mathrm{E}\{y \mid y \in Y,\ f^{-1}y \subseteq A\}$$

will be called the *small image* of $A$.

**Definition.** A multivalued mapping $f : X \to Y$ will be called a *preclosed mapping* if for every point $y$ of $Y$ and for every neighbourhood $Of^{-1}y$ of its large inverse image $f^{-1}y$, there exists a set $H$ such that $f^{-1}y \subseteq H \subseteq Of^{-1}y$ and that the large image $fH$ of $H$ is an open set in $Y$.

Remarks. 1. The set of all interior points of a set $M$ is called the interior of $M$ and denoted by Int $M$. It is easy to see that $f : X \to Y$ is preclosed if and only if for each point $y \in Y$ and for each neighbourhood $Of^{-1}y$, we have $y \in \mathrm{Int}\, f(Of^{-1}y)$.

2. $f : X \to Y$ is said to be closed (open) if $fA$ is closed (open) for every closed (open) set $A \subseteq X$. A moment's consideration shows that any closed (open) mapping is a preclosed mapping.

In our papar [4] we have proved some theorems about univalued preclosed mappings. We shall mention here some interesting results ($f : X \mapsto Y$ denotes a univalued continuous mapping):

1. Let $f : X \mapsto Y$ be a preclosed, monotone[1]) mapping, and let $A$ be a set such that $A = f^{-1}fA$. Then if $\dim A = 0$ we have $\dim fA = 0$, if $\operatorname{ind} A = 0$ then $\operatorname{ind} fA = 0$, and if $\operatorname{Ind} A = 0$ we have $\operatorname{Ind} fA = 0$.

2. Let $f : X \mapsto Y$ be a preclosed, bicompact mapping, and let $\omega R$ denote the weight of the space $R$. Then we have $\omega Y \leqq \omega X$.

3. Let $X$ and $Y$ be Hausdorff spaces, $aX$ — an extension of $X$, $cY$ — a perfect extension of $Y$ (we use here the term due to E. G. Sklyarenko [2]). Let further $f : X \mapsto Y$ be a preclosed mapping, which has an extension to a perfect (i.e., a closed, bicompact, continuous) mapping $f_{ac} : aX \mapsto cY$.

If $f$ is a monotone[1]) mapping, then $f_{ac}$ is also a monotone mapping.

We want to give some results concerning multivalued preclosed mappings. We have

**Lemma 1.** *Let $f : X \to Y$ be a multivalued mapping, and let $G$ be an open-closed subset of $X$. If $f$ is a monotone*[1]) *mapping, then the large image of $G$ coincides with the small image $fG = f_b G$. If $f$ is a monotone and preclosed mapping, then this image $fG$ is also an open-closed set (of $Y$).*

**Theorem 1.** *Let $f : X \to Y$ be a monotone and preclosed mapping. If $Y$ is a connected space, then $X$ is also a connected space.*

Now let two inifinite regular cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ be given, $\mathfrak{a} \leqq \mathfrak{b}$. A set $M$ is said to be an $[\mathfrak{a}, \mathfrak{b}]$-*compact set* if from any open covering $\gamma$ of $M$, which has the power $\overline{\overline{\gamma}} = \mathfrak{m} \leqq \mathfrak{b}$, we can choose a subcovering $\gamma'$, the power of which $\overline{\overline{\gamma'}} = \mathfrak{k} < \mathfrak{a}$. The notion of $[\mathfrak{a}, \mathfrak{b}]$-compactness has been defined by P. S. Alexandroff and P. S. Urysohn. The characterization, which we use here, is due to Yu. M. Smirnov.

A set $M$ is said to be an $[\mathfrak{a}, \infty]$-*compact set* if it is an $[\mathfrak{a}, \mathfrak{b}]$-compact set for every $\mathfrak{b}$.

We shall say that a set $M$ is a *locally* $[\mathfrak{a}, \mathfrak{b}]$-*compact set* if its every point has a neighbourhood $U_x$ the closure $\overline{U}_x$ of which is an $[\mathfrak{a}, \mathfrak{b}]$-compact set.

$f : X \to Y$ is said to be an $[\mathfrak{a}, \mathfrak{b}]$-*compact* ($[\mathfrak{a}, \infty]$-*compact*) *mapping* if the large inverse image $f^{-1}y$ of every point $y \in Y$ is an $[\mathfrak{a}, \mathfrak{b}]$-compact ($[\mathfrak{a}, \infty]$-compact) set.

Finally, we shall say that $f : X \to Y$ is *strongly continuous* if the inverse mapping $f^{-1}$ is both open and closed.

We have

**Theorem 2.** *Let $f$ be a strongly continuous, preclosed, $[\mathfrak{a}, \infty]$-compact mapping from a space $X$ onto a regular space $Y$. Then the local $[\mathfrak{a}, \mathfrak{b}]$-compactness will be preserved.*

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[1]) $f : X \mapsto Y$ ($f : X \to Y$) is said to be monotone, if the (large) inverse image of every point $y$ of $Y$ is a connected set.

**Theorem 3.** *Let $f : X \to Y$ be a preclosed, $[\mathfrak{a}, \mathfrak{b}]$-compact mapping and let $Y_1$ be an $[\mathfrak{a}, \infty]$-compact subset of Y. Then the large inverse image $X_1 = f^{-1}Y_1$ is an $[\mathfrak{a}, \mathfrak{b}]$-compact set.*

Remark. In the case of univalued mappings, this theorem is, in a certain sense, the generalization of a theorem, due to Yu. M. Smirnov (v. [3], theorem 5).

### References

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