Jun-iti Nagata

A survey of dimension theory

In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium, 1966. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967. pp. 259--270.

Persistent URL: http://dml.cz/dmlcz/700895

A SURVEY OF DIMENSION THEORY

JUN-ITI NAGATA

Pittsburgh

The purpose of this lecture is to give a survey of modern dimension theory emphasizing the development since 1961 when the first Prague Symposium was held. As for the development before 1961 I should like to invite people's attention to the excellent surveys by P. Alexandroff [1], [2], [3], [4]. (The last paper covers results after 1961, too, emphasizing Soviet mathematicians' works.)

The lecture will be divided into two parts, dimension theory of metric spaces and that for non-metrizable spaces. Generally speaking we have a very well-established dimension theory for metric spaces though new topics in this aspect still do not seem to be exhausted. On the other hand we are not yet in a position to be satisfied with the present status of dimension theory for non-metrizable spaces.

1. Theory for metric spaces

Since M. Katětov and K. Morita extended principal results of the classical dimension theory like sum theorem, decomposition theorem and product theorem to general metric spaces and proved $\dim R = \operatorname{Ind} R$ for every metric space R, there has been a remarkable progress in the theory for metric spaces. We dare say we know more about dimension of general metric spaces than they did of separable metric spaces twenty years ago. ($\dim R$, $\operatorname{Ind} R$ and $\operatorname{ind} R$ denote covering (or Lebesgue), large inductive (or Čech) and small inductive (or Urysohn-Menger) dimension respectively. As for basic definitions and theorems in dimension theory J. Nagata [1] is recommended as a reference.)

One of the most important events since 1961 was the negative settlement of the famous question "$\dim R = \operatorname{ind} R$ for every metric space R?" by P. Roy [1]. He presented a *complete metric space* R *with* $\operatorname{ind} R = 0$, $\operatorname{Ind} R = \dim R = 1$ which is a set of sequences of real numbers with a complicated topology. On the other hand it is well known (See A. Zarelua [1]) that

Theorem. $\operatorname{ind} R = \operatorname{Ind} R = \dim R$ *for every metric space* R *which is a sum of countably many closed strongly metrizable spaces, i.e. metric spaces with* σ*-star-finite open bases.* (Let us temporarily call such a space an S-space.)

Although it will be possible to sligthly extend this coincidence theorem, at present there seems to be no very interesting category of metric spaces between S-spaces and general metric spaces. Thus we can say we now have a quite good information about the condition for ind R and Ind R (= dim R) to coincide. But still some questions, for example, "For any integers n, m with $0 \leqq n < m$ is there a metric space R of ind $R = n$, dim R = Ind $R = m$?", remain open as pointed out by R. Hodel. Besides, the theory on ind R of general metric spaces will be of some interest because it can differ from dim R in such an important space like a complete metric space. For example it will be an interesting problem to find a universal metric space of ind $R \leqq n$.

As for covering dimension J. Nagata [2] obtained a universal n-dimensional space as follows:

Theorem. *Denote by $P(A)$ the product of countably many star-spaces with index A where $|A| = \tau$ and by $K_n(A)$ the set of points in $P(A)$ at most n of whose non-vanishing coordinates are rational. Then a metric space R has* dim $\leqq n$ *and weight $\leqq \tau$ iff it is homeomorphic to a subset of $K_n(A)$. We get an imbedding theorem for countable-dimensional spaces (= a countable union of* 0*-dimensional sets) replacing the words 'at most n' with 'at most finitely many'.*

(Let $\{I_\alpha \mid \alpha \in A\}$ be a system of unit segments $[0, 1]$. By identifying all zeros in $\bigcup\{I_\alpha \mid \alpha \in A\}$ we get a star-shaped set $S(A)$. Defining a metric ϱ in $S(A)$ by

$$\varrho(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same segment } I_\alpha, \\ x + y & \text{if } x, y \text{ belong to distinct segments,} \end{cases}$$

we obtain a metric space called star-space with index A.)

Comparing the general imbedding theorem with the classical one for separable metric spaces we notice that $P(A)$ has infinite dimension while every n-dimensional separable metric space is imbedded in the $(2n+1)$-dimensional Euclidean cube I^{2n+1}. This leads us to the following problem "Improve the general imbedding theorem finding another universal n-dimensional set in a (simple) finite-dimensional space instead of in $P(A)$". Any product of finitely many star-spaces does not serve this purpose, because it is an S-space while not every finite-dimensional metric space is an S-space as pointed out by Yu. Smirnov. Thus we have another question "Can we find a universal n-dimensional S-space in a product of finitely many star-spaces?" Moreover the latter part of the above imbedding theorem leads us to the following conjecture. "The set of points in $P(A)$ at most finitely many of whose coordinates are non-vanishing is a universal space for strongly countable-dimensional spaces (= a countable union of finite-dimensional closed subsets)?" It is also unknown whether the homeomorphic mappings from an n-dimensional metric space R into $K_n(A)$ are dense in the metric space $C(R, P(A))$ of all continuous mappings from R into $P(A)$ as in the separable case (C. Kuratowski's problem).

The negative answer to the long unsolved L. Tumarkin's problem by D. Henderson [1] was another remarkable event. Namely he constructed an *infinite-dimensional compact metric space Q in a Hilbert cube such that Q has no n-dimensional closed subsets for* $1 \leqq n < \infty$. The space Q is not a countable-dimensional space because if it were, then it would have a transfinite inductive dimension (since Q is compact metric), and thus contain closed subsets of every finite dimension. However it is not yet known whether Q is weakly infinite-dimensional in the meaning of P. Alexandroff (for any countable number of pairs of disjoint closed sets (C_i, C_i'), $i = 1, 2, \ldots$, there are closed sets B_i, $i = 1, 2, \ldots$ which separate C_i from C_i' and satisfy $\bigcap_{i=1}^{\infty} B_i = \emptyset$). If it is proved to be true, then it negatively solves another famous question of P. Alexandroff "Does countable-dimensionality coincide with weak infinite-dimensionality for any compact metric space?" Henderson also posed an interesting conjecture "Every strongly infinite-dimensional (= not weakly infinite-dimensional) compact metric space contains a compactum with no positive-dimensional subcompacta?"

As seen in Henderson's result infinite-dimensional spaces have many negative properties. Here is another example due to K. Nagami and J. Roberts [1] of the negativity.

Theorem. *The set K_ω of points in a Hilbert cube, whose coordinates are zero except finitely many at the most, has no metric completion which is countable-dimensional.*

It is well known that K_ω is a universal space for strongly countable-dimensional separable metric spaces (and therefore it is countable-dimensional of course) and that every finite-dimensional metric space has a completion with the same dimension even if it is not separable. We owe another interesting result in the field of infinite dimension theory to A. Arhangelski [1] who proved

Theorem. *Every metric space R has an increasing sequence $\{J_i \mid i < \omega_1\}$ of 0-dimensional subsets such that $\bigcup\{J_i \mid i < \omega_1\} = R$.*

K. Nagami [1] has shown that if the space R is countable-dimensional, then ω_1 can be replaced with ω_0, which is a generalization of Tumarkin's result. However the main purpose of the Arhangelski's paper is to develop the idea of rank of covering which was given by Nagata at the first Prague Symposium (See J. Nagata [1]). Arhangelski got several interesting results in this aspect, among which are

Theorem. *A normal space R has* dim $\leqq n$ *iff for every finite open covering $\mathfrak{U}$, there exists an open refinement $\mathfrak{V}$ of rank $\leqq n + 1$.*

Theorem. *A compact T_1-space which has a base of rank 1 is metrizable.*

But the converse is not true. As a matter of fact he proved

Theorem. *A metric space R is strongly countable-dimensional iff it has a basis $\mathfrak{U}$ such that $\operatorname{rank}_x \mathfrak{U} < +\infty$ at every point x of R.*

These theorems lead us to a speculation that it might be of some interest to investigate general topological spaces which have a base $\mathfrak{U}$ such that $\operatorname{rank}_x \mathfrak{U} < +\infty$ at every point x (or $\operatorname{rank} \mathfrak{U} < +\infty$).

Let us turn our attention to topics on the relation between dimension and metric function. At the first Prague Symposium I posed the following conjecture:

Theorem. *A metric space R has $\dim \leqq n$ iff it admits a metric ϱ such that for every $n + 3$ points $x, y_1, \ldots, y_{n+2}$ in R there is a pair of indices i, j satisfying*

$$\varrho(y_i, y_j) \leqq \varrho(x, y_j) \quad (i \neq j)\,.$$

The reason why I call it a theorem is that this conjecture was proved to be true by J. Nagata [3] and P. Ostrand [1]. To prove it Ostrand used the following theorem, interesting in itself:

Theorem. *A metric space R has $\dim \leqq n$ iff for each open covering $\mathfrak{U}$ of R and each integer $k \geqq n + 1$, there exist k discrete families $\mathfrak{U}_1, \ldots, \mathfrak{U}_k$ of open sets such that the union of any $n + 1$ of the $\mathfrak{U}_i$ is a covering of R which refines $\mathfrak{U}$.*

He not only proved the above conjecture by this theorem but also applied it to prove the following remarkable theorem (P. Ostrand [2]) which is a generalization of A. Kolmogoroff and V. Arnold's theorem answering Hilbert's problem 13 in the negative:

Theorem. *For $p = 1, 2, \ldots, m$, let R_p be a compact metric space of finite dimension d_p and let $n = \sum_{p=1}^{m} d_p$. Then there exist continuous functions ψ^{pq} from R_p into $[0, 1]$ for $p = 1, \ldots, m$ and $q = 1, \ldots, 2n + 1$ such that every continuous, real function f defined on $\prod_{p=1}^{m} R_p$ is represented in the form*

$$f(x_1, \ldots, x_m) = \sum_{q=1}^{2n+1} \varphi_q\Big[\sum_{p=1}^{m} \psi^{pq}(x_p)\Big]\,,$$

where the functions φ_q are real and continuous.

Various modifications and generalizations may possibly stem from this theorem. Aside from the dimension theory it will be an interesting problem to seek conditions for every continuous function defined on a topological product to be expressible as (or approximable by) a continuous function of continuous functions on the coordinate spaces. On the other hand the following J. de Groot's conjecture, which is similar to the solved conjecture, still remains open: "A metric space R has $\dim \leqq n$ iff it admits a metric ϱ such that for every $n + 3$ points $x, y_1, \ldots, y_{n+2}$ in R there is a triplet of indices i, j, k satisfying

$$\varrho(y_i, y_j) \leqq \varrho(x, y_k) \quad (i \neq j)\text{?"}$$

A metric of such a special type was used by K. Nagami [2] to reprove Katětov-Morita's theorem: Ind $R = \dim R$ for every metric space R. It will be worthwhile to try to extend Nagata-Ostrand's theorem to the infinite-dimensional case. Namely, "Is it true that a metric space R is countable-dimensional (or strongly countable-dimensional) iff it admits a metric ϱ such that for every sequence $x, y_1, y_2, \ldots$ of points in R there is a pair of indices i, j satisfying $\varrho(y_i, y_j) \leqq \varrho(x, y_j)$ $(i \neq j)$? If not, then what kind of space is the space which allows such a metric?"

We cannot pass the works on metric dependent dimension functions by K. Nagami, J. Roberts and his students either. For example, K. Nagami and J. Roberts [2] studied relations between dimension functions $d_2, d_3, d_4, \mu \dim$ (= metric dimension). The function d_2 is defined as $d_2(\emptyset) = -1$, $d_2(R) \leqq n$ if for any $n+1$ pairs of closed sets (C_i, C_i'), $i = 1, \ldots, n+1$ with $d(C_i, C_i') > 0$ there are closed sets B_i, $i = 1, \ldots, n+1$ which separate C_i and C_i' for each i and $\bigcap_{i=1}^{n+1} B_i = \emptyset$. Replacing the word '$n+1$ pairs' with 'finite number of pairs' ('countable number of pairs') they define $d_3(d_4)$. Then

Theorem. $d_2(R) \leqq d_3(R) \leqq \mu \dim R \leqq d_4(R) = \dim R \leqq 2\mu \dim R$ *for any metric space R.* (The last part of the inequality is the well-known Katětov's theorem.)

They pose an interesting question "$d_3(R) = \mu \dim R$ for any metric space?" after proving it for totally bounded spaces. In this connection we should note that the recently published book of J. R. Isbell [1] contains the result of his extensive study on uniform dimension.

Now, let us concern ourselves with the field of mapping and dimension. (All mappings concerned are continuous.) Here we already have a quite well-established theory (especially for closed mappings), but various interesting results are still being obtained. One of them is the following theorem of A. Zarelua and Yu. Smirnov [1] which has a combined form of Alexandroff's theorem and Hurewicz's theorem.

Theorem. *A compact metric space R has exactly dimension n iff there is an essential* 0*-dimensional mapping of R into the n-dimensional cube* I^n.

They also gave a similar characterization in a non-metrizable case with the use of decomposing mapping which is a new generalization of Katětov's uniformly 0-dimensional mapping. Recently open mappings have been more seriously studied than ever. For example R. Hodel [1] developed Alexandroff and Robert's works in this aspect. He especially obtained the following generalization of Alexandroff's theorem:

Theorem. *Let f be an open at most countable-to-one mapping from a locally compact metric space R onto a metric space S. Then* $\dim R = \dim S$.

(The same theorem is true for paracompact spaces R and S if the condition 'countable-to-one' is replaced with 'finite-to-one' as shown by Nagami.) We shall return to dimension and mapping later in the non-metrizable case.

J. de Groot and T. Nishiura [1] studied dimension of compactifications. Namely, comp $R \leqq n$ is inductively defined in a similar way as ind $R \leqq n$ but beginning with comp $R = -1$ for any compact R, and def R = the least n such that R has a compactification S for which $\dim (S - R) = n$. They obtained relations between comp R, def R and dim R for separable metric spaces R pursuing the similarity of theory on comp R to dimension theory. De Groot's conjecture "comp $R \leqq n$ iff def $R \leqq n$?" still remains open though it is known to be true for $n = -1, 0$ and for some other special cases.

Let me conclude this section with a special but interesting result of A. Mischenko [1]:

Theorem. *Every metric space R can be imbedded into a homogeneous metric space $G(R)$ with* $\dim G(R) = \dim R$.

As a matter of fact $G(R)$ is a free group generated by R with a left invariant metric which is an extension of the metric of R.

2. Theory for non-metrizable spaces

In contrast to the metric case, there are many fundamental problems to be solved in the non-metrizable case. For example, it is not known whether ind R = Ind R for every compact space R though it is not true for normal spaces. (As a matter of fact K. Nagami [3] constructed a normal space Z of ind $Z = 0$, dim $Z = 1$, Ind $Z = 2$.) It also remains unknown whether $\dim R \leqq \operatorname{Ind} R$ for every paracompact space (Throughout this section we assume every space is at least Hausdorff though some discussions may remain true without the Hausdorff condition. Moreover, 'n-dimensional' means covering dimension $= n$ unless the contrary is explicitly mentioned.) We may risk saying that the only very satisfactory result in the non-metrizable case (in comparison with the metric case) is that of the sum theorems which are established both for dim R (R : normal) and Ind R (R : hereditarily paracompact) in a general form by K. Morita, M. Katětov, C. H. Dowker, A. Zarelua and others (See J. Nagata [1]). However remarkable progress has been achieved recently in various aspects, and it makes a complete survey extremely difficult.

Among the most remarkable results on relations between different dimension functions is B. Pasynkov's [1]

Theorem. $\dim R = \operatorname{ind} R = \operatorname{Ind} R = \operatorname{ind} G - \operatorname{ind} H$ *for every factor space $R = G/H$ of a locally compact group G by a closed subgroup H.*

K. Nagami [4], too, obtained a similar result and also proved that

Theorem. *Every locally compact group G with* $\dim G = n$ *can be decomposed into the sum of $n + 1$ 0-dimensional paracompact subspaces B_i, $i = 1, \ldots, n + 1$.*

As implied by the works of Mischenko, Pasynkov and Nagami, it is a quite interesting and prospective problem to study dimension in connection with the homogeneity of the space. (More generally speaking, homogeneity has never been thoroughly investigated from the point of view of general topology.) A. Zarelua [2] and V. Ponomarev [1] also worked hard to obtain interesting conditions for non-metrizable spaces under which different dimension functions coincide.

In the above papers of Pasynkov, Nagami and Ponomarev the method of inverse system (spectra) was shown to be quite useful. The study of this interesting device itself has also proved to be very fruitful, which leads us to expect that it may soon become a central topic in dimension theory. The following result of S. Mardešić [1], which is a generalization of Freudental's classical theorem, as is well known, made an epoch in this aspect of study:

Theorem. *Every compact space of* dim $\leqq n$ *is the inverse limit of an inverse system of compact metric spaces of* dim $\leqq n$.

B. Pasynkov [2] generalized this theorem as follows:

Theorem. 1. *Every n-dimensional paracompact space is the limit of an inverse system of n-dimensional metric spaces.*

2. *Every n-dimensional regular Lindelöf space is the limit of an inverse system of n-dimensional separable metric spaces.*

3. *A strongly paracompact space R has* dim $\leqq n$ *iff R is the limit of an inverse system of n-dimensional metric spaces.*

Recently V. Kljušin [1] added more to our knowledge in this aspect proving

Theorem. *Every n-dimensional paracompact M-space (paracompact topologically complete space) is the limit of an inverse system of n-dimensional metric spaces (complete metric spaces) and of perfect projection mappings.*

(A topological space R is called an M-space if there is a closed continuous mapping from R onto a metric space S such that $f^{-1}(q)$ is countably compact for every $q \in S$.)

In view of the active study of inverse limit, I wonder if it is not worthwhile to investigate dimension in relation to another similar concept inductive limit. (See, for example, J. Dugundji [1].)

One of the most significant developments in general dimension theory is the recent construction of universal n-dimensional spaces for completely regular spaces by B. Pasynkov [3] and A. Zarelua [2]. (Probably these are the first decisive results in this aspect.) Answering a question of Alexandroff they independently succeeded to give *an n-dimensional compact space* $P_{n\tau}$ *with weight* τ *such that every completely regular space of* dim $\leqq n$ *and weight* $\leqq \tau$ *is homeomorphic to a subset of* $P_{n\tau}$. The methods of Pasynkov and Zarelua are each quite different from the other but the former's method is more elementary. Letting $\{R_\alpha \mid \alpha \in A\}$ be the collection of all completely regular spaces of dim $\leqq n$ and weight $\leqq \tau$, Pasynkov constructed his universal space as a continuous image of $\beta(\bigcup\{\beta R_\alpha \mid \alpha \in A\})$, where β denotes Stone-Čech compactification while the union is a discrete union. His result also has a relation with Mardešić's result mentioned above, because both are proved by use of Mardešić's theorem on mapping factorization. Pasynkov also showed that his method simply reformulates Nagata's universal n-dimensional metric space (in a generalized Hilbert space) when applied on metric spaces.

The method of inverse limit was shown again to be powerful when B. Pasynkov [4] used it to prove

Theorem. *Every compact space S of weight τ is the image of some compact space R of* dim $\leqq 1$ *and weight τ under an open 0-dimensional mapping.*

In the field of dimension and mapping we should also note the following theorem of E. Sklyarenko [1].

Theorem. *Let f be a closed mapping of a paracompact space R onto a paracompact space S. If* $\dim f^{-1}(q) \leqq n$ *for every $q \in S$, then* $\dim R \leqq \dim S + n$.

Although the same theorem is well-known for metric spaces, it has never been proved before for non-metrizable spaces in such a beautiful form.

There was a substantial progress in the field of infinite dimension theory, too. (As for the references to the development of infinite dimension theory around 1960, Yu. Smirnov's [1] brief but excellent survey should be noted in addition to Alexandroff's survey papers mentioned at the beginning.) For example E. Sklyarenko [2], [3] obtained the following results.

Theorem. *Let $B = G/H$ be the factor space of an infinite-dimensional locally compact group G by a closed subgroup H. If B is infinite-dimensional, then it is strongly infinite-dimensional and contains Hilbert cube I^ω.*

Theorem. *A compact space R is strongly infinite-dimensional iff there is a mapping f of R into I^ω such that for each finite-dimensional face F of I^ω the mapping $f: f^{-1}(F) \to F$ is essential.*

Both are especially interesting in the meaning that they deal with the comparatively unknown areas 'dimension and homogeneity' and 'characterization of grade of infinite-dimensionality by mapping'.

B. Levšenko [1] proved the following theorem with respect to transfinite large inductive dimension.

Theorem. *Let R be a hereditarily normal space such that $R = R_1 \cup R_2$,* $\operatorname{Ind} R_1 = \lambda + p$, $\operatorname{Ind} R_2 = \mu + q$, *where λ, μ are limit ordinal numbers while p, q are integers. Then*

$$\operatorname{Ind} R \leqq \max(\lambda + p, \mu + q) \quad \text{if} \quad \lambda \neq \mu,$$

$$\operatorname{Ind} R \leqq \lambda + p + q + 1 \quad \text{if} \quad \lambda = \mu.$$

In the field of dimension of compactification Yu. Smirnov [2] approached to a problem similar to the one considered by de Groot and Nishiura from a different direction, analogy with covering dimension. Namely he proved

Theorem. $\dim(\beta R - R) = \dim^{\infty} R$ *if R is a normal space and any two disjoint closed sets of $\beta R - R$ can be separated by open sets in βR, where* $\dim^{\infty} R \leqq n$ *means that any finite open collection* $\{U_1, \ldots, U_k\}$ *of R for which* $R - \bigcup_{i=1}^{k} U_i$ *is compact is refined by a finite open collection with the same property and order* $\leqq n+1$.

It is natural to return to the following basic questions before discussing a satisfactory dimension theory to be established in non-metrizable spaces.

1. What is the most adequate dimension function (for the purpose of dimension theory in non-metrizable spaces)?
2. What is the best place (space)?
3. What theorems (at least) should be contained in the theory?

As for the first question our first concern naturally goes to $\dim R$ and $\operatorname{Ind} R$ though any possibility of other new dimension functions should not be excluded. Concerning 2, I have some doubt about the adequacy of general normal spaces or even of general paracompact spaces though many good theorems are being proved for them. At any rate I hope a prospective general dimension theory to cover a category of spaces which includes all metric spaces and compact spaces as special cases, because we already have a very good dimension theory for metric spaces and a considerably good one for compact spaces. In this respect a dimension theory for paracompact M-spaces may be a good first step to begin with. (Every metric space and compact space are paracompact M-spaces, and the countable product of paracompact M-spaces is paracompact M.) As for question 3, we now have good sum theorems and a considerably good theory for dimension and mapping, so product theorem may be a next keypoint to be overcome.

Before introducing this problem let me mention the P-space of K. Morita [1] which is doubtless one of the most splendid products of general topology in these years. A topological space R is called a P-space if for any collection $\{G(\alpha_1 \ldots \alpha_i) \mid \alpha_1, \ldots, \alpha_i \in \Omega,\ i = 1, 2, \ldots\}$ of open sets of R such that

$$G(\alpha_1 \ldots \alpha_i) \subset G(\alpha_1 \ldots \alpha_i \alpha_{i+1}),$$

there is a collection $\{F(\alpha_1 \ldots \alpha_i) \mid \alpha_1, \ldots, \alpha_i \in \Omega,\ i = 1, 2, \ldots\}$ of closed sets of R satisfying

i) $F(\alpha_1 \ldots \alpha_i) \subset G(\alpha_1 \ldots \alpha_i)$,

ii) if $R = \bigcup_{i=1}^{\infty} G(\alpha_1 \ldots \alpha_i)$ for a sequence $\{\alpha_1, \alpha_2, \ldots\}$, then $R = \bigcup_{i=1}^{\infty} F(\alpha_1 \ldots \alpha_i)$.

Morita proved that for the product $R \times S$ to be normal for any metrizable space S it is necessary and sufficient that R be a normal P-space, and that every M-space is a P-space. It seems that there is a sort of parallelism between product theorems in general topology ($R \times S$ is normal, paracompact etc.) and product theorems in dimension theory ($\dim R \times S \leqq \dim R + \dim S$, $\operatorname{Ind} R \times S \leqq \operatorname{Ind} R + \operatorname{Ind} S$, etc.) and therefore that not only M-space but P-space also has some importance in the dimension theory. Now, let us cite cases in which the product theorem on dimension is proved to be true.

Theorem. $\dim R \times S \leqq \dim R + \dim S$

in each of the following cases.

1. *R is paracompact and a countable union of locally compact closed sets, and S is paracompact* (K. Morita [2]. *This theorem implies that if R is a CW-complex, then the equality* $\dim R \times S = \dim R + \dim S$ *holds.*)
2. *S is separable metric, and $R \times S$ is normal and countably paracompact.* (N. Kimura [1].)
3. *R is normal P, and S is strongly metrizable* (N. Kimura [2]).

The new theorem 3 of Kimura leads us to the question "What is the necessary and sufficient condition for a normal P-space R in order that $R \times S$ be a normal space of dim $\leqq n + m$ for any metric space S of dim $\leqq m$?" It will not be so difficult to solve the question in the case of $n = m = 0$ modifying Morita's theory on P-spaces, but the general case will be worth a serious investigation.

Theorem. $\operatorname{Ind} R \times S \leqq \operatorname{Ind} R + \operatorname{Ind} S$

in each of the following cases:

1. *R is a totally normal space (each open subspace of R has a locally finite open covering by open sets which are F_σ in R) such that the product space $R \times S$ is totally normal for any metrizable space S, and S is metrizable.* (K. Morita [3]. *This includes the case in which R is perfectly normal, and S is metrizable; this improves Nagami's product theorem which requires R to be perfectly normal and paracompact.*)
2. *R and S are paracompact M, and $R \times S$ is totally normal.*

Although the condition in 2 can be slightly weakened it is not known whether we can remove the condition for $R \times S$ from 2 or whether we can replace the condition M for the space R in 2 with P. It is also open (though quite likely) whether 2 is true for covering dimension (presumably without the condition for $R \times S$).

Let me conclude my lecture with a (prejudiced) list of grade to show how good (or bad) the present status of dimension theory for non-metrizable spaces is in comparison with the theory for (separable and non-separable) metric spaces.

Dimension theory for	metric spaces	non-metrizable spaces
Sum theorem	E	E
Product theorem	E	F–G
Decomposition theorem	E	F
Imbedding theorem	G	F
Subspace theorem	E	F–G
Dimension and mapping	E	G
Dimension and inverse limit	G	F–G
Dimension and metric function	G	—
Infinite dimension theory	G	F
Relation between different dimensions	E	F–G

E = excellent, G = good, F = fair.

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