Toposym 2

Jerzy Mioduszewski; L. Rudolf

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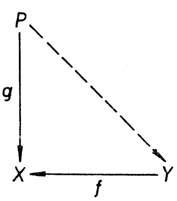
A FORMAL CONNECTION BETWEEN PROJECTIVENESS FOR COMPACT AND NOT NECESSARILY COMPACT COMPLETELY REGULAR SPACES

J. MIODUSZEWSKI and L. RUDOLF

Wrocław

The purpose of this note is to give a formal connection between the notions of projective spaces and projective resolutions for compact and non-compact completely regular spaces. The compact case was considered by Gleason [2] and Rainwater [7], and the non-compact one by Iliadis [5], Ponomarev [6] and Flachsmeyer [1] (let us remark that some results of these papers are valid for Hausdorff or regular spaces, too; in [5] and [6] there was considered in fact a notion of "absolute space" closely related to the projectiveness). We show that the general case reduces to the compact one.

Let $\boldsymbol{C}$ be a category. An object P is said to be *projective with respect to a class* $\boldsymbol{A}$ of morphisms of the category iff for each $f \in \boldsymbol{A}$ and each g in $\boldsymbol{C}$ there exists a morphism in $\boldsymbol{C}$ completing the diagram

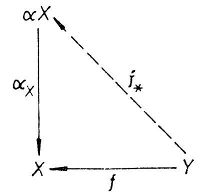


In the category of compact completely regular spaces *projective spaces* are assumed to be projective objects with respect to the class $\boldsymbol{A}$ of all mappings onto. The class of all projective spaces of this category is equal to each of the class of spaces:

1. all extremally disconnected spaces,
2. all spaces each mapping onto which is a retraction.

The first assertion is known e.g. from [2], the proof of the second one is purely formal.

By a *projective resolution with respect to* $\boldsymbol{A}$ of an object X of the category $\boldsymbol{C}$ we mean a projective object αX and a morphism $\alpha_X : \alpha X \to X$ from $\boldsymbol{A}$, which is *irreducible with respect to* $\boldsymbol{A}$, i.e. for no proper subobject S of αX the composite morphism $S \to \alpha X \to X$ belongs to $\boldsymbol{A}$. If a morphism $p : Z \to X$ from $\boldsymbol{A}$ is a *semi-monomorphism*, i.e. the equality $p \circ q = p$ implies that p is the identity, and Z is projective, then p is irreducible; such a semimonomorphism $p : Z \to X$ is unique for X up to an isomorphism. Clearly, for each morphism $f : Y \to X$, where Y is projective, there exists a morphism f_* completing the diagram



In [2] and [7] the existence of projective resolutions is showed for the category of compact completely regular spaces; in [7] the uniqueness follows from the fact that α_X is a semimonomorphism. The meaning of subobject is a compact (= closed) subspace. The problem whether f_* is uniquely determined by f, i.e. that of the functorial character of operation α, was considered by Henriksen and Jerison [4]. The answer is "yes" iff mapping f is such that $\overline{\mathrm{Int} f^{-1}(A)} = \overline{f^{-1}(\mathrm{Int}\, A)}$ for each regularly closed subset A of X.

In the category of completely regular spaces, in order to exclude the triviality (= discreteness), *projective spaces* are assumed to be projective objects with respect to the class A of all *perfect* mappings onto, i.e. closed mappings onto, for which all inverse images of points are compact. Perfect mappings are characterized by Henriksen and Isbell [3] as such mappings $f : Y \xrightarrow{\text{onto}} X$ for which the induced mappings $\beta f : \beta Y \to \beta X$ in the Čech-Stone compactification transform $\beta Y - Y$ into (in fact, onto) $\beta X - X$. In the proofs of theorems which follow only this formal property of perfect mappings is used.

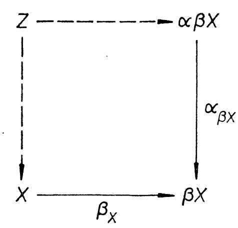
The class of all projective spaces just defined is equal to each of the class of spaces:

1'. all extremally disconnected spaces,

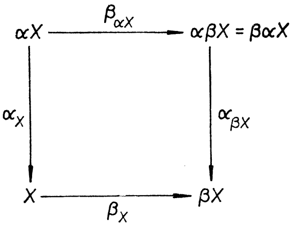
2'. all spaces each perfect mapping onto which is a retraction

(the first assertion is known from [1]).

Now, if X is a completely regular space, we construct αX, the projective resolution of X, taking firstly βX and $\alpha\beta X$, the projective resolution of βX, and then the uniformization (pullback diagram)



of the Čech-Stone embedding β_X and of the projective covering $\alpha_{\beta X}$. Mapping $Z \to X$ is the desired projective resolution $\alpha_X : \alpha X \to X$ of X. The proof of this fact reduces to the showing that Z is projective (= extremally disconnected) and that $Z \dot{\to} X$ is a perfect semimonomorphism onto. By the way, $Z \to \alpha\beta X$ occurs to be the Čech-Stone embedding. Hence, the operations α and β commute. More precisely, the diagram



commutes (see [5]).

If we restrict the considerations of mappings $f: Y \to X$ to the class considered in [4], the operation α is functorial, and only in this case.

The proofs will be published in Colloquium Mathematicum.

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