BAIRE SETS WHICH ARE BORELIAN SUBSPACES

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J. Knowles and C. Rogers [2] recognized the good behaviour of Borelian subspaces which are Baire sets; these sets are called BB-sets here (while the term descriptive Baire sets was used in [2]). They proved the characterization (2) of BB's in Theorem B below, and used that characterization in developing of properties of BB's following the patterns used in [1] or [3] for Borelian spaces; the resulting proofs were rather complicated.

Here we want to indicate a smooth development of properties of BB's. Theorems B(2), C and D were proved in [2]. The complete proofs (with Theorems E and D in a more general setting) with appear in Proc. Roy. Soc. A (1967) under a similar title.

By a space we always mean a uniformizable (=completely regular) topological space, and $\sum$ will stand for the space of irrationals. A disjoint ($\{fx \mid x \in P\}$ is disjoint) upper semicontinuous compact (the sets fx are compact) multivalued mapping of P into Q will be called a DUCC mapping. By [1, Theorem 7] a space P is Borelian iff there exists a DUCC mapping of $\sum$ on P (Borelian spaces were introduced and studied in [1]).

Theorem A. *Each of the following conditions is necessary and sufficient for a Borelian subspace X of a space P to be a Baire set in P:*

(a) *There exists a countable collection $\mathscr{F}$ of continuous functions on P which distinguishes the points of X from the points of $P - X$ (i.e., if $x \in X$, $y \in P - X$ then $fx \neq fy$ for some f in $\mathscr{F}$).*

(b) *There exists a continuous mapping f of P into a separable metrizable space such that $f[X] \cap f[P - X] = \emptyset$.*

(c) *If k is* DUCC *mapping of $\sum$ onto X then there exists an f as in* (b) *such that $f \circ k$ is disjoint (and so a* DUCC *mapping).*

(d) *There exists a* DUCC *mapping of $\sum$ onto X and an f as in* (b) *such that $f \circ k$ is disjoint.*

(e) *Condition* (b) *with $f[X]$ a Baire set, in the range space of f.*

Theorem B. *Each of the following two conditions is necessary and sufficient for a subset X of P to be a* BB-*set in P:*

(1) *X is Borelian, and if k is a* DUCC *mapping of $\sum$ onto X then there exists a continuous mapping F of $\sum$ into $C^*(P)$ such that kx is the zero set of Fx for each x in $\sum$.*

(2) *There exist k and F with properties stated in* (1).

Theorem C. *Every Baire set in a* BB-*set in P is a* BB-*set in P. The* BB'*s of P form a σ-ring.*

Theorem D. *Every Borelian subspace of P is a Baire set provided that every closed set is a G_δ. In particular, in a perfectly normal space every Borelian subspace is a Baire set.*

The main step is the following Separation Theorem.

Theorem E. *If X is an analytic subspace of a space P and if Y is an F_σ in P disjoint to X, then there exists a Baire set Z in P with $X \subset Z$, $Z \cap Y = \emptyset$.*

References

[1] *Z. Frolík:* A contribution to the descriptive theory of sets and spaces. General Topology and its Relations to Modern Analysis and Algebra. (Proceedings of the Symposium held in Prague in September, 1961), 157—173.

[2] *J. Knowles and C. Rogers:* Descriptive sets. Proc. Roy. Soc. A *291* (1966), 353—367.

[3] *C. Rogers:* Descriptive Borel sets. Proc. Roy. Soc. A *286* (1965), 455—478.

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