Andrzej Lelek
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In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium, 1966. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967. pp. 239--240.

Persistent URL: http://dml.cz/dmlcz/700847

ON QUASI-COMPONENTS

A. LELEK

Wrocław

The *quasi-component* $Q(X, x)$ of a space X at a point $x \in X$ is the intersection of all closed-open subsets of X that contain x. Let us write $Q^0(X, x) = X$ and use a transfinite induction to define $Q^\alpha(X, x)$ for each ordinal α, namely

$$Q^{\alpha+1}(X, x) = Q(Q^\alpha(X, x), x)$$

and

$$Q^\lambda(X, x) = \bigcap_{\alpha<\lambda} Q^\alpha(X, x)$$

for limit λ. We call $Q^\alpha(X, x)$ the *quasi-component of order* α of the space X at the point x. Thus quasi-components are quasi-components of order 1.

Let Ω denote the least uncountable ordinal, and consider a space X which has a countable open basis. Since the decreasing sequence

$$Q^0(X, x) \supset Q^1(X, x) \supset \ldots \supset Q^\alpha(X, x) \supset \ldots$$

consists of closed subsets of X, there is an ordinal $\beta < \Omega$ such that $Q^\beta(X, x) = Q^{\beta+1}(X, x)$. The ordinal

$$nc(X, x) = \min \{\beta : Q^\beta(X, x) = Q^{\beta+1}(X, x)\}$$

is called the *non-connectivity index* of the space X at the point x.

Let $\boldsymbol{B}_j^k(X)$ denote the j-th Borel class ($j = 0, 1, \ldots$), additive when $k = 0$, and multiplicative when $k = 1$, of subsets of X. Thus, for instance, the elements of $\boldsymbol{B}_1^0(X)$ are all F_σ-sets in X and the elements of $\boldsymbol{B}_1^1(X)$ are all G_δ-sets in X.

Theorem 1. *If P is the pseudo-arc and $p \in P$, then for every ordinal $\alpha < \Omega$ there exists a set $P_\alpha \subset P$ such that*

$$p \in P_\alpha \in \boldsymbol{B}_1^0(P) \cap \boldsymbol{B}_1^1(P) \quad \textit{and} \quad nc(P_\alpha, p) = \alpha\,.$$

Given any collection $\boldsymbol{C}$ of subsets of a space X, we say a set U is *componentwise universal* in $\boldsymbol{C}$ provided $U \in \boldsymbol{C}$ and there exists a closed subset Y of X such that each set $C \in \boldsymbol{C}$ is homeomorphic to a set $U \cap V$, where V is a component of Y.

Theorem 2. *If P is the pseudo-arc, then there exists a componentwise universal set in each Borel class* $\boldsymbol{B}_j^k(P)$.

The following result is a consequence of Theorem 1.

Theorem 3. *If a compact metric space X contains the pseudo-arc and a set U is componentwise universal in a Borel class $\boldsymbol{B}_j^k(X)$, where $j > 0$, then the non-connectivity index of U is unbounded, i.e.,*

$$\Omega = \sup \{nc(U, u) : u \in U\} .$$

The first example of a space with unbounded non-connectivity index was constructed by Taĭmanov [2]. His example was a G_δ-set in the Euclidean 3-space, and he attributed to P. S. Novikov the problem whether or not there exists such a set on the plane. It follows from Theorems 2 and 3 that the pseudo-arc contains a G_δ-set, as well as an F_σ-set, with unbounded non-connectivity index.

However, in all those examples one has already uncountably many quasi-components of order 1. This suggests the following question. Does there exist a separable metric space with unbounded non-connectivity index and such that for each $\alpha < \Omega$ the collection of all quasi-components of orders less than α is countable? In other words, can a separable metric space have uncountably many quasi-components of higher orders but only countably many quasi-components of orders bounded by any countable ordinal? The question seems to be related to an example of partially ordered set, given by Specker [1].

The proofs of Theorems 1—3 will be published in a paper on the topology of curves, to appear in Fundamenta Mathematicae.

References

[1] *E. Specker:* Sur un problème de Sikorski. Colloq. Math. *2* (1951), 9—12.
[2] *А. Д. Тайманов:* О квазикомпонентах несвязных множеств. Мат. сб. *25* (1949), 367—386.