THE COMPACTNESS OPERATOR IN GENERAL TOPOLOGY

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The role of (bi)compactness has increased tremendously during the last half century. This abstract indicates a further strengthening of this notion (at the expense of the Hausdorff property, e.g.).

Let X be a set, and $\mathscr{F}$ a family of subsets of X. Let ε denote the operator, which assigns to $\mathscr{F}$ the collection $\varepsilon(\mathscr{F})$, that is the family of all finite unions and arbitrary intersections of members from $\mathscr{F}$. We do not assume that $\varepsilon(\mathscr{F})$ necessarily contains $\emptyset$ and X as elements.

Such a family $\varepsilon(\mathscr{F})$ on a set X is called a minus-topology $(X, \varepsilon(\mathscr{F}))$ over X. (It can, of course, always be extended to a topology over X.)

A subset S of X is called compact relative to $\mathscr{F}$, as ususal, provided that every subfamily $\mathscr{F}'$ of $\mathscr{F}$, for which $\mathscr{F}' \cup \{S\}$ has the finite intersection property, has a non empty intersection in S. So, to any $\mathscr{F}$ corresponds a family of compact sets $\varrho(\mathscr{F})$ in $(X, \varepsilon(\mathscr{F}))$, where ϱ is called the compactness operator.

The elements of $\varrho(\varrho(\mathscr{F})) = \varrho^2(\mathscr{F})$ are called square-compact subsets of $(X, \varepsilon(\mathscr{F}))$. A subset S of X is apparently square-compact, if every subcollection $(\varrho(\mathscr{F}))'$ of $\varrho(\mathscr{F})$ for which $(\varrho(\mathscr{F}))' \cup \{S\}$ has the finite intersection property, has a non empty intersection in S. We call $\varrho^2 = \sigma$ the square-compactness operator.

We have the following connections between these operators.

(1) $\varrho\varepsilon = \varrho$.

Observe that (1) is a reformulation of Alexander's Lemma!

(2) $\varepsilon\sigma = \sigma\varepsilon = \sigma$;

(3) $\varepsilon^2 = \varepsilon$; $\sigma^2 = \sigma$.

For the proof of the propositions (2) and (3) we need a lemma.

Lemma. *Let C be a subset of X and an element of $\varrho(\mathscr{F})$; let E be a subset of X and an element of $\varrho^2(\mathscr{F})$. Then $C \cap E$ is an element of $\varrho^2(\mathscr{F}) \cap \varrho(\mathscr{F})$.*

Proof. a) Let $\mathscr{C}'$ be a sub-collection of $\varrho(\mathscr{F})$ such that $\mathscr{C}' \cup \{C \cap E\}$ has the finite intersection property. Then $\mathscr{C}' \cup \{C\} \cup \{E\}$ has the finite intersection property (further written f.i.p.). But since $\mathscr{C}' \cup \{C\} \subset \varrho(\mathscr{F})$ and $E \in \varrho^2(\mathscr{F})$ we have $(\bigcap\mathscr{C}') \cap \cap C \cap E \neq \emptyset$ which proves that $C \cap E \in \varrho^2(\mathscr{F})$.

b) Choose $\mathscr{F}' \subset \mathscr{F}$ such that $\mathscr{F}' \cup \{C \cap E\}$ has the finite intersection property. Then the collection $\mathscr{F}'' = \{F \cap C \mid F \in \mathscr{F}'\}$ has also the finite intersection property in E.

It is obvious that the elements of $\mathscr{F}''$ are compact relative to $\mathscr{F}$, because each element is an intersection of a subbasic closed set and a compact set. Hence $\mathscr{F}''$ is a subcollection of $\varrho(\mathscr{F})$ with the finite intersection property in E and consequently $(\bigcap\mathscr{F}'') \cap E$ is non empty. From this we obtain $(\bigcap\mathscr{F}') \cap (C \cap E) \neq \emptyset$, thus $(C \cap E) \in \in \varrho(\mathscr{F})$.

Proposition (2). *The collection $\varrho^2(\mathscr{F}) = \sigma(\mathscr{F})$ is closed under finite unions and arbitrary intersections.*

Proof. The fact that $\varrho^2(\mathscr{F})$ is closed under finite unions is a consequence of the definition of $\varrho^2(\mathscr{F})$. Now we will prove that $\varrho^2(\mathscr{F})$ is closed under arbitrary intersections.

Consider a collection $\mathscr{E}' \subset \varrho^2(\mathscr{F})$ such that $\bigcap\mathscr{E}' = E_0 \neq \emptyset$, (the case that $\bigcap\mathscr{E}' = = \emptyset$ is trivial).

We must prove that every collection $\mathscr{C}'$, such that $\mathscr{C}' \cup \{E_0\}$ has the f.i.p., has a non empty intersection in E_0.

Pick and fix a member $E_1 \in \mathscr{E}'$ and consider the collection $\mathscr{C}'' = \{C \cap E \mid C \in \mathscr{C}'; E \in \mathscr{E}'\}$.

From the Lemma it follows that the members of $\mathscr{C}''$ are members of $\varrho(\mathscr{F})$. By assumption $\mathscr{C}'' \cup \{E_1\}$ has the f.i.p. and hence $(\bigcap\mathscr{C}'') \cap E_1 \neq \emptyset$; but this intersection equals $(\bigcap\mathscr{C}') \cap E_0$ and this proves that $E_0 = (\bigcap\mathscr{E}') \in \varrho^2(\mathscr{F})$.

Proposition (3). $\varrho^2(\mathscr{F}) = \varrho^4(\mathscr{F})$.

Proof. We first prove that $\varrho(\mathscr{F}) \subset \varrho^3(\mathscr{F})$. Let C be an element of $\varrho(\mathscr{F})$ and let $\mathscr{E}'$ be a subcollection of $\varrho^2(\mathscr{F})$ such that $\mathscr{E}' \cup \{C\}$ has the f.i.p.

Pick and fix some $E_0 \in \mathscr{E}'$ and consider $\tilde{\mathscr{C}} = \{C \cap E \mid E \in \mathscr{E}'\}$.

From the Lemma it follows that each member of $\tilde{\mathscr{C}}$ is a member of $\varrho(\mathscr{F})$ and clearly $\tilde{\mathscr{C}} \cup \{E_0\}$ has the f.i.p.

Thus $(\bigcap\tilde{\mathscr{C}}) \cap E_0 \neq \emptyset$; $C \cap (\bigcap\mathscr{E}') \neq \emptyset$ and hence C is a member of $\varrho^3(\mathscr{F})$, which proves that $\varrho(\mathscr{F}) \subset \varrho^3(\mathscr{F})$.

Similarly we can find that $\varrho^2(\mathscr{F}) \subset \varrho^4(\mathscr{F})$.

On the other hand $\varrho^2(\mathscr{F})$ is defined as being the collection of compact sets relative to $\varrho(\mathscr{F})$ and $\varrho^4(\mathscr{F})$ as being the collection of compact sets relative to $\varrho^3(\mathscr{F})$. From $\varrho(\mathscr{F}) \subset \varrho^3(\mathscr{F})$ it follows that $\varrho^2(\mathscr{F}) \supset \varrho^4(\mathscr{F})$.

Hence $\varrho^2(\mathscr{F}) = \varrho^4(\mathscr{F})$.

$\varepsilon\sigma = \sigma$ says that for every $\mathscr{F}$ the family $\varrho^2(\mathscr{F})$ forms a minus topology on X.

The second part of (3) tells us in particular that the ϱ operator is "of finite order" and the relations (2) and (3) determine the structure of the semigroup $\{\varepsilon, \sigma\}$; ε is an identity, and σ is an idempotent.

Let us discuss now a few special cases of importance.

I. $\varrho = \varepsilon$ holds exactly for those topological spaces in which the compact sets coincide with the closed sets. The results above become trivial.

II. $\varrho^2 = \varepsilon$. In this case ϱ and ε form a group of order 2 with ε as the identity. This case has been studied in [1]. Spaces supplied with such a minus topology are called antispaces. These are exactly those spaces in which the square-compact subsets coincide with the closed subsets. The locally compact Hausdorff spaces and the metrizable spaces are e.g. antispaces.

If $(X, \mathscr{G})$ is an antispace with a minus topology, then also $(X, \varrho(\mathscr{G}))$ is an antispace with a minus topology. $(X, \mathscr{G})$ and $(X, \varrho(\mathscr{G}))$ determine themselves mutually.

In particular, if $(X, \mathscr{G})$ is the real line R, then $(X, \varrho(\mathscr{G}))$ is an antispace and the corresponding topology gives us a compact non-Hausdorff T_1 space, denoted by ϱR, and a large part of mathematics could be based onto ϱR instead of R, since $\varrho^2 R = R$.

Reference

[1] *J. de Groot:* An isomorphism criterium in general topology (1966). Bull. Am. Math. Soc. *73* (1967).