SYMMETRIC APPROACH TO THE FUNDAMENTAL NOTIONS OF GENERAL TOPOLOGY

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1. The notions $\mathfrak{B}$, $\mathbf{C}$, $\mathscr{L}$, $\mathscr{A}$. BIRKHOFF was the first to consider the interrelations between different fundamental notions of set-theoretic topology as Galois correspondences (Galois connexions) in the sense of ORE. In this communication, we are going to deal with the notions

$\mathfrak{B}$ like voisinages (neighbourhoods), historically connected with the name of HAUSDORFF;

$\mathbf{C}$ like closure, connected with the name of KURATOWSKI;

$\mathscr{L}$ like limit, connected with the name of FRÉCHET;

$\mathscr{A}$ like adhérence (set of cluster points) as considered by BOURBAKI, but as it seems without connection with any classical author (the first published study of $\mathscr{A}$ as a fundamental notion seems to be by GRIMEISEN [11]).

To be more precise, we consider some fundamental set, the carrier or space X. Then $\mathfrak{B}$ may be considered as a — in the first stage quite arbitrary — binary relation between X and power set $\mathfrak{P}(X)$, $\mathfrak{B} \subset X \times \mathfrak{P}(X)$. So according to the scheme of general binary relations, with each point $x \in X$ is associated its neighbourhood system, i.e. the set

$$\mathfrak{B}x = \{A \mid (x, A) \in \mathfrak{B}\}$$

of all point sets $A \subset X$ such that the couple (x, A) is in the relation $\mathfrak{B}$; so the latter, i.e. $A \in \mathfrak{B}x$, may be read "A is a neighbourhood of x". Naturally, $\mathfrak{B}$ might as well be thought of as some onevalued mapping $\mathfrak{B} : X \to \mathfrak{P}(\mathfrak{P}(X))$. On the other hand, the interpretation of $\mathfrak{B}$ as a binary relation enables us to form the converse relation $\mathfrak{B}^{-1} \subset \mathfrak{P}(X) \times X$,

$$(A, x) \in \mathfrak{B}^{-1} \quad \text{iff} \quad (x, A) \in \mathfrak{B}$$

so that there is associated with point set $A \subset X$ the point set

$$\mathfrak{B}^{-1}A = \{x \mid (x, A) \in \mathfrak{B}\}$$

which might be called the interior of A; accordingly $x \in \mathfrak{B}^{-1}A$ might be read "x is an interior point of A".

Second, we may consider $\mathbf{C}$ as a — in the first stage quite arbitrary — binary relation of the same type as $\mathfrak{B}^{-1}$, i.e. $\mathbf{C} \subset \mathfrak{P}(X) \times X$, so that with each point set

$B \subset X$ is also associated the point set

$$\mathbf{C}B = \{x \mid (B, x) \in \mathbf{C}\}$$

usually called the closure (abgeschlossene Hülle, adhérence) of B, consisting of all points x such that the couple (B, x) is in the relation $\mathbf{C}$. For the latter, i.e. $x \in \mathbf{C}B$, unfortunately there is no generally accepted terminology in the English literature; in German, $x \in \mathbf{C}B$ is read "x ist Berührungspunkt von B" (ALEXANDROFF-HOPF) in French, "x est point adhérent à B" (BOURBAKI). Besides there are well-known non-topological situations, when one reads $x \in \mathbf{C}B$ or $B\,\mathbf{C}\,x$ as "x depends on B". Again, as is most frequently done in topology (but curiously enough not in abstract dependence theory!), one may also consider as some one-valued mapping, the closure operator $\mathbf{C} : \mathfrak{P}(X) \to \mathfrak{P}(X)$. Still the relational point of view enables us to form the converse relation $\mathbf{C}^{-1} \subset X \times \mathfrak{P}(X)$, of the same type as $\mathfrak{B}$, and so to associate with each point $x \in X$ the set

$$\mathbf{C}^{-1}x = \{B \mid (B, x) \in \mathbf{C}\}$$

of all point sets $B \subset X$ which might be said to "touch" ("berühren" in German) x, or to adhere to x such that $\mathbf{C}^{-1}x$ might be called the adherence system associated with point x.

Third, $\mathscr{L}$ may be considered as a – in the first stage quite arbitrary – binary relation between the set $\Phi(X)$ of all filters on X and space X itself, $\mathscr{L} \subset \Phi(X) \times X$, so that with each filter (or equivalently – for those prefering a more analytical language and not minding a more complicated technique – : net) $\mathfrak{B}$ is associated a point set

$$\mathscr{L}\mathfrak{B} = \{x \mid (\mathfrak{B}, x) \in \mathscr{L}\}$$

which might be called the limit set of $\mathfrak{B}$ since its points x are usually called the limit points (Grenzwerte, points limites) of $\mathfrak{B}$, the converse relation, $\mathfrak{B} \in \mathscr{L}^{-1}x$, being read as "$\mathfrak{B}$ converges to x". Let us remember that in more general situations, namely in the topological theory of nets of points sets (instead of nets of points), $\mathscr{L}\mathfrak{B}$ is most frequently spoken of as the lower or inferior limit (Limes inferior, ensemble-limite inférieure).

Last, $\mathscr{A}$ may be considered as a – in the first stage quite arbitrary – binary relation of the same type as $\mathscr{L}$, $\mathscr{A} \subset \Phi(X) \times X$, the set

$$\mathscr{A}\mathfrak{B} = \{x \mid (\mathfrak{B}, x) \in \mathscr{A}\}$$

being called the upper or superior limit (Limes superior or Adhärenz, ensemble-limite supérieure or adhérence), its points the cluster or accumulation points (Häufungspunkte, Berührungspunkte, points adhérents) of filter $\mathfrak{B}$.

2. Transitions between these notions are established by the following 12 statements holding as elementary propositions in general topology:

$(\mathfrak{B} - \mathbf{C})$	$x \in \mathfrak{B}^{-1}A \Leftrightarrow \bigwedge\limits_{B}(x \in \mathbf{C}B \Rightarrow A * B)$	$x \in \mathbf{C}B \Leftrightarrow \bigwedge\limits_{A}(x \in \mathfrak{B}^{-1}A \Rightarrow A * B)$
$(\mathfrak{B} - \mathscr{A})$	$x \in \mathfrak{B}^{-1}A \Leftrightarrow \bigwedge\limits_{\mathfrak{B}}(x \in \mathscr{A}\mathfrak{B} \Rightarrow A * \mathfrak{B})$	$x \in \mathscr{A}\mathfrak{B} \Leftrightarrow \bigwedge\limits_{A}(x \in \mathfrak{B}^{-1}A \Rightarrow A * \mathfrak{B})$
$(\mathfrak{B} - \mathscr{L})$	$x \in \mathfrak{B}^{-1}A \Leftrightarrow \bigwedge\limits_{\mathfrak{B}}(x \in \mathscr{L}\mathfrak{B} \Rightarrow A \in \mathfrak{B})$	$x \in \mathscr{L}\mathfrak{B} \Leftrightarrow \bigwedge\limits_{A}(x \in \mathfrak{B}^{-1}A \Rightarrow A \in \mathfrak{B})$
$(\mathbf{C} - \mathscr{A})$	$x \in \mathbf{C}A \Leftrightarrow \bigvee\limits_{\mathfrak{B}}(x \in \mathscr{A}\mathfrak{B} \wedge A \in \mathfrak{B})$	$x \in \mathscr{A}\mathfrak{B} \Leftrightarrow \bigwedge\limits_{A}(x \in \mathbf{C}A \Leftarrow A \in \mathfrak{B})$
$(\mathbf{C} - \mathscr{L})$	$x \in \mathbf{C}A \Leftrightarrow \bigvee\limits_{\mathfrak{B}}(x \in \mathscr{L}\mathfrak{B} \wedge A * \mathfrak{B})$	$x \in \mathscr{L}\mathfrak{B} \Leftrightarrow \bigwedge\limits_{A}(x \in \mathbf{C}A \Leftarrow A * \mathfrak{B})$
$(\mathscr{A} - \mathscr{L})$	$x \in \mathscr{A}\mathfrak{A} \Leftrightarrow \bigvee\limits_{\mathfrak{B}}(x \in \mathscr{L}\mathfrak{B} \wedge \mathfrak{A} * \mathfrak{B})$	$x \in \mathscr{L}\mathfrak{B} \Leftrightarrow \bigwedge\limits_{\mathfrak{A}}(x \in \mathscr{A}\mathfrak{A} \Leftarrow \mathfrak{A} * \mathfrak{B})$

Here $A * B$ means "set A intersects set B", i.e. $A \cap B \neq \emptyset$. $A * \mathfrak{B}$ means "set A intersects filter $\mathfrak{B}$", i.e. $A * B$ for all sets $B \in \mathfrak{B}$, classically: "(infinitely) many members of net (sequence) $\mathfrak{B}$ are in set A" or "have property A", "net $\mathfrak{B}$ is frequently in A". Dually, $A \in \mathfrak{B}$ has the classical meaning and may be read: "almost all members of net (sequence) $\mathfrak{B}$ are in set A" or "have property A", "net $\mathfrak{B}$ is eventually in A". Finally, $\mathfrak{A} * \mathfrak{B}$ means "filter $\mathfrak{A}$ intersects filter $\mathfrak{B}$", i.e. $A * B$ for all sets $A \in \mathfrak{A}$, $B \in \mathfrak{B}$; this is what has been called "compatible" by SAMUEL, "compositive" by SMILEY (who had taken this term from the unknown work of E. H. MOORE). In the complete lattice $\Phi(X)$ of all filters including the improper filter, i.e. the full power set $\mathfrak{P}(X)$, as lattice unit, $\mathfrak{A} * \mathfrak{B}$ means that the supremum of filters (sum of dual ideals) $\mathfrak{A}, \mathfrak{B}$ is a proper filter, i.e. unequal to $\mathfrak{P}(X)$. Remembering that sets $A \subset X$ are in a natural one-to-one correspondence with principal filters $[\![A, X]\!] = \{F \mid A \subset F \subset X\}$, we may consider this compatibility relation $*$ as an extension from the domain of principal filters to that of all filters, principal or not. Still it should be emphasized that within this wider domain, relation $*$ has not been used in classical analysis; $\mathfrak{A} * \mathfrak{B}$ may be understood as "nets (sequences) $\mathfrak{A}, \mathfrak{B}$ have a common refinement" or — in a fairly wide sense — "a common subnet (subsequence)". In fact, classical analysis carefully avoided this relation between nets (sequences) by replacing the two statements $(\mathscr{A} - \mathscr{L})$ by the usual formulas

$$(1) \qquad x \in \mathscr{A}\mathfrak{A} \Leftrightarrow \bigvee_{\mathfrak{B}}(x \in \mathscr{L}\mathfrak{B} \wedge \mathfrak{A} \subset \mathfrak{B}), \quad x \in \mathscr{L}\mathfrak{B} \Leftrightarrow \bigwedge_{\mathfrak{A}}(x \in \mathscr{A}\mathfrak{A} \Leftarrow \mathfrak{A} \supset \mathfrak{B});$$

x is a cluster point of sequence $\mathfrak{A}$ iff there is a subsequence $\mathfrak{B}$ converging to x, x is a limit of sequence $\mathfrak{B}$ iff x is a cluster point of all subsequences $\mathfrak{A}$. Under self-evident monotony assumptions on $\mathscr{A}$ and $\mathscr{L}$, these two classical statements are equivalent with $(\mathscr{A} - \mathscr{L})$ of our list; but it is by this very list, by its intrinsic analogies, symmetries, and dualities that the naturalness of the relation $\mathfrak{A} * \mathfrak{B}$, instead of the classical relations $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \supset \mathfrak{B}$, is emphasized. What is most striking: the classical formulas (1) do not describe a Galois correspondence as we shall see $(\mathscr{A} - \mathscr{L})$ does, since in both formulas of $(\mathscr{A} - \mathscr{L})$ the same relation $\mathfrak{A} * \mathfrak{B}$ occurs, whereas in the formulas (1), we really have two different relations, $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \supset \mathfrak{B}$.

Besides, extending relation $*$ still further from filters to quite arbitrary set-systems (sets of sets), we may reformulate $(\mathfrak{B} - \mathbf{C})$

$$A \in \mathfrak{B}x \Leftrightarrow A * \mathbf{C}^{-1}x\,, \quad B \in \mathbf{C}^{-1}x \Leftrightarrow B * \mathfrak{B}x\,, \tag{2}$$

or in the shortest possible notation

$$\mathfrak{B}x = (\mathbf{C}^{-1}x)^*\,, \quad \mathbf{C}^{-1}x = (\mathfrak{B}x)^*\,, \tag{3}$$

where in general $\mathfrak{S}^*$ denotes the system of sets intersecting system $\mathfrak{S}$. By the last formulation, it becomes particularly suggestive that with each point x there are associated really two systems, one being the $*$-system of the other: so HAUSDORFF's and KURATOWSKI's classical approaches appear as – in a strict sense – dual to each other.

There are also more concentrated formulations for the other statements of our list. For instance, the second formulas of $(\mathfrak{B} - \mathscr{A})$ and $(\mathfrak{B} - \mathscr{L})$ may be written

$$x \in \mathscr{A}\mathfrak{B} \Leftrightarrow \mathfrak{B}x * \mathfrak{B}\,, \tag{4}$$

$$x \in \mathscr{L}\mathfrak{B} \Leftrightarrow \mathfrak{B}x \subset \mathfrak{B}\,. \tag{5}$$

In particular, if $\mathfrak{B}x$ were a filter, the neighbourhood filter of x, it would become the smallest filter – a representative of the coarsest net converging to x.

Further abbreviations in the form of set-theoretic identities:

$$\mathfrak{B}x = \bigcap_{x \in \mathscr{A}\mathfrak{B}} \mathfrak{B}^*\,, \tag{6}$$

$$\mathfrak{B}x = \bigcap_{x \in \mathscr{L}\mathfrak{B}} \mathfrak{B}\,; \tag{7}$$

$$\mathbf{C}A = \bigcup_{A \in \mathfrak{B}} \mathscr{A}\mathfrak{B}\,, \quad \mathscr{A}\mathfrak{B} = \bigcap_{A \in \mathfrak{B}} \mathbf{C}A\,, \tag{8}$$

$$\mathbf{C}A = \bigcup_{A * \mathfrak{B}} \mathscr{L}\mathfrak{B}\,, \quad \mathscr{L}\mathfrak{B} = \bigcap_{A * \mathfrak{B}} \mathbf{C}A \tag{9}$$

$$\mathscr{A}\mathfrak{A} = \bigcup_{\mathfrak{A} * \mathfrak{B}} \mathscr{L}\mathfrak{B}\,, \quad \mathscr{L}\mathfrak{B} = \bigcap_{\mathfrak{A} * \mathfrak{B}} \mathscr{A}\mathfrak{A}\,. \tag{10}$$

Again, under a weak monotony assumption on $\mathscr{A}$, the first formula of $(\mathbf{C} - \mathscr{A})$ is equivalent with

$$\mathbf{C}A = \mathscr{A}[\![A, X]\!]\,, \tag{11}$$

so the adhérence of set A appears to be nothing but the adhérence of the associated principal filter $[\![A, X]\!]$, and operator $\mathscr{A}$: $\Phi(X) \to \mathfrak{P}(X)$ as an extension of operator $\mathbf{C} : \mathfrak{P}(X) \to \mathfrak{P}(X)$. Finally, under the dual monotony assumption on $\mathscr{L}$, the $*$ in the first formula of (9) might be replaced by $\in$ (as has been usually done in classical analysis); yet this would be perfectly impossible in the second formula: otherwise – as comparison of (8) and (9) shows – no distinction would be left between $\mathscr{A}$ and $\mathscr{L}$.

3. Pairs of transitions considered as Galois correspondences. Returning to our original list of transition formulas, we may say that they define 12 possible maps between the sets of all $\mathfrak{V}$, $\mathbf{C}$, $\mathscr{A}$, and $\mathscr{L}$ respectively. So $(\mathfrak{V} - \mathbf{C})$ associates with each arbitrary neighbourhood relation $\mathfrak{V}$ a definite closure (or dependence) relation $\mathbf{C} = \mathbf{C}_{\mathfrak{V}}$, and with each arbitrary closure relation $\mathbf{C}$ a definite neighbourhood relation $\mathfrak{V} = \mathfrak{V}_{\mathbf{C}}$, etc. Now the couples of transition mappings

$$\mathbf{C} \mapsto \mathfrak{V}_{\mathbf{C}}, \quad \mathfrak{V} \mapsto \mathbf{C}_{\mathfrak{V}},$$
$$\mathscr{A} \mapsto \mathfrak{V}_{\mathscr{A}}, \quad \mathfrak{V} \mapsto \mathscr{A}_{\mathfrak{V}},$$
$$\mathscr{L} \mapsto \mathfrak{V}_{\mathscr{L}}, \quad \mathfrak{V} \mapsto \mathscr{L}_{\mathfrak{V}}$$

as defined in $(\mathfrak{V} - \mathbf{C})$, $(\mathfrak{V} - \mathscr{A})$, and $(\mathfrak{V} - \mathscr{L})$ are Galois correspondences in the sense of ORE: they are anti-monotone with respect to inclusion (remember that $\mathfrak{V}$ $\mathbf{C}$, $\mathscr{A}$, and $\mathscr{L}$, as binary relations, are subsets of some cartesian products, so inclusion, i.e. the usual comparison of binary relations, makes sense); moreover the twofold mappings, e.g.

$$\mathbf{C} \mapsto \mathbf{C}_{\mathfrak{V}_{\mathbf{C}}}, \quad \mathfrak{V} \mapsto \mathfrak{V}_{\mathbf{C}_{\mathfrak{V}}}$$

are closure operators. Analogously, $(\mathbf{C} - \mathscr{A})$, $(\mathbf{C} - \mathscr{L})$, and $(\mathscr{A} - \mathscr{L})$ define couples of mappings which are Galois correspondences in some dual sense called of mixed type in [18, II]: the simple mappings themselves, e.g.

$$\mathscr{A} \mapsto \mathbf{C}_{\mathscr{A}}, \quad \mathbf{C} \mapsto \mathscr{A}_{\mathbf{C}},$$

are monotone with respect to inclusion, and of the twofold mappings

$$\mathscr{A} \mapsto \mathscr{A}_{\mathbf{C}_{\mathscr{A}}}, \quad \mathbf{C} \mapsto \mathbf{C}_{\mathscr{A}_{\mathbf{C}}},$$

the first is again a closure operator, whereas the second is of dual type, a kernel operator (monotone, idempotent, but intensive, $\mathbf{C}_{\mathscr{A}_{\mathbf{C}}} \subset \mathbf{C}$, not extensive).

4. The main questions about these Galois correspondences are answered by

Theorem.

$\mathbf{C} = \mathbf{C}_{\mathfrak{V}_{\mathbf{C}}} \Leftrightarrow \mathbf{C}$ monotone $\mathscr{A} = \mathscr{A}_{\mathfrak{V}_{\mathscr{A}}} \Leftrightarrow \mathscr{A}$ continuous from below $\mathscr{L} = \mathscr{L}_{\mathfrak{V}_{\mathscr{L}}} \Leftrightarrow \mathscr{L}$ pretopological	$\mathfrak{V} = \mathfrak{V}_{\mathbf{C}_{\mathfrak{V}}} \Leftrightarrow \mathfrak{V}$ monotone $\mathfrak{V} = \mathfrak{V}_{\mathscr{A}_{\mathfrak{V}}} \Leftrightarrow \mathfrak{V}$ monotone $\mathfrak{V} = \mathfrak{V}_{\mathscr{L}_{\mathfrak{V}}} \Leftrightarrow \mathfrak{V}$ pretopological
$\mathscr{A} = \mathscr{A}_{\mathbf{C}_{\mathscr{A}}} \Leftrightarrow \mathscr{A}$ continuous from below $\mathscr{L} = \mathscr{L}_{\mathbf{C}_{\mathscr{L}}} \Leftrightarrow \mathscr{L}$ pretopological $\mathscr{L} = \mathscr{L}_{\mathscr{A}_{\mathscr{L}}} \Leftrightarrow \mathscr{L}$ pseudo-topological	$\mathbf{C} = \mathbf{C}_{\mathscr{A}_{\mathbf{C}}} \Leftrightarrow \mathbf{C}$ monotone $\mathbf{C} = \mathbf{C}_{\mathscr{L}_{\mathbf{C}}} \Leftrightarrow \mathbf{C}$ pretopological $\mathscr{A} = \mathscr{A}_{\mathscr{L}_{\mathscr{A}}} \Leftrightarrow \mathscr{A}$ pseudo-topological

Here *monotony* of $\mathfrak{V}$ means that, for each point x, system $\mathfrak{V}x$ is monotone, i.e. $A \subset A'(\subset X) \wedge A \in \mathfrak{V}x \Rightarrow A' \in \mathfrak{V}x$. Analogously, *monotony* of $\mathbf{C}$ means that, for each point x, system $\mathbf{C}^{-1}x$ is monotone, equivalently: that operator $\mathbf{C} : \mathfrak{P}(X) \to \mathfrak{P}(X)$ is monotone in the usual sense $B \subset B'(\subset X) \Rightarrow \mathbf{C}B \subset \mathbf{C}B'$ (the equivalence of this condition with $\mathbf{C} = \mathbf{C}_{\mathfrak{V}_{\mathbf{C}}}$ going back to MARKOFF). That $\mathfrak{V}$ is *pretopological* means that, for each point x, $\mathfrak{V}x$ is a filter (proper or not) on X. Dually, that $\mathbf{C}$ is *pretopological* means that, for each point x, $\mathbf{C}^{-1}x$ is a grillage (French: grille) as defined by CHOQUET, i.e. just the *-system of a filter; or in better known terms: that operator $\mathbf{C} : \mathfrak{P}(X) \to \mathfrak{P}(X)$ preserves finite (including the empty) unions: $\mathbf{C}(B_1 \cup B_2) = \mathbf{C}B_1 \cup \mathbf{C}B_2$, $\mathbf{C}\emptyset = \emptyset$.

For the description of the properties of $\mathscr{A}$ and $\mathscr{L}$ as quoted in the Theorem, let us remember the following two fundamental composition laws (cf. [18, I], also DIENER [7]) in the complete lattice $\Phi(X)$ of all filters on X:

($\uparrow$) the composition from below: each filter $\mathfrak{F} \in \Phi(X)$ is the lattice-supremum, even the set-theoretic union of an upwards directed set of principal filters $\mathfrak{P}$, the latter being precisely the finitely generated filters or, in lattice-terms, the inaccessible filters, i.e. those which cannot be represented as suprema of upward directed sets of other filters, those which are indecomposable, irreducible, or prime with respect to this composition from below;

($\downarrow$) the composition from above: each filter $\mathfrak{F} \in \Phi(X)$ is the lattice-infimum, i.e. the set-theoretic intersection of a set of ultrafilters $\mathfrak{A}$, the latter being precisely the totally meet-irreducible filters, i.e. those which cannot be represented as meets (lattice-infima) of any sets of other filters, those indecomposable with respect to this composition from above.

As we have seen, $\mathscr{A}$ may be considered as an extension of $\mathbf{C}$ from the domain of principal filters $\mathfrak{P}$ (the prime elements of the composition from below) to arbitrary filters $\mathfrak{F}$. Now, the *continuity from below* as quoted in our Theorem demands that this extension is not arbitrary, but

$$\mathscr{A} = \underline{\mathscr{A}}\,, \tag{12}$$

where by definition

$$\underline{\mathscr{A}}\mathfrak{F} := \bigcap_{\mathfrak{P} \subset \mathfrak{F}} \mathscr{A}\mathfrak{P} = \bigcap_{P \in \mathfrak{F}} \mathscr{A}[\![P, X]\!]\,. \tag{13}$$

Let us remark that in general

$$\mathscr{A}, \underline{\mathscr{A}} \subset \mathscr{A}_{\mathbf{C}_{\mathscr{A}}} = \mathscr{A}_{\mathfrak{V}_{\mathscr{A}}}\,. \tag{14}$$

Our Theorem states that continuity from below, $\mathscr{A} = \underline{\mathscr{A}}$, is equivalent with $\mathscr{A} = \mathscr{A}_{\mathbf{C}_{\mathscr{A}}}$, more explicitly

$$\mathscr{A}\mathfrak{F} = \bigcap_{G \in \mathfrak{F}} \bigcup_{G \in \mathfrak{H}} \mathscr{A}\mathfrak{H} \tag{15}$$

(which condition has been considered by GRIMEISEN). Finally, a nearly immediate equivalence for $\mathscr{A} = \underline{\mathscr{A}}$ is the equation

$$\mathscr{A} \bigcup_{\mathfrak{G}\in\Gamma} \mathfrak{G} = \bigcap_{\mathfrak{G}\in\Gamma} \mathscr{A}\mathfrak{G} \quad \text{(for any upwards directed } \Gamma \subset \Phi(X)) . \tag{16}$$

Associated with the composition from above, there is the *continuity from above,*

$$\mathscr{A} = \overline{\mathscr{A}} \tag{17}$$

where by definition

$$\overline{\mathscr{A}}\mathfrak{F} := \bigcup_{\mathfrak{U}\supset\mathfrak{F}} \mathscr{A}\mathfrak{U} = \bigcup_{\substack{\mathfrak{G}\supset\mathfrak{F} \\ 0\notin\mathfrak{G}}} \bigcap_{\substack{\mathfrak{H}\supset\mathfrak{G} \\ 0\notin\mathfrak{H}}} \mathscr{A}\mathfrak{H} . \tag{18}$$

By definition, these continuous from above operators $\mathscr{A} : \Phi(X) \to \mathfrak{P}(X)$ are in one-to-one correspondence with their restrictions to the domain $\Omega(X)$ of ultrafilters $\mathfrak{U}$, these restrictions $\mathscr{A}|\Omega(X) : \Omega(X) \to \mathfrak{P}(X)$ (due to the incomparability of ultrafilters) being completely arbitrary mappings, called pseudo-topologies by CHOQUET [6]; hence we may call any continuous from above $\mathscr{A}$ also *pseudo-topological* as in our Theorem. Let us remark that in general

$$\mathscr{A}, \overline{\mathscr{A}} \supset \mathscr{A}_{\mathscr{L}_{\mathscr{A}}} \tag{19}$$

Our Theorem states that continuity from above, $\mathscr{A} = \overline{\mathscr{A}}$, is equivalent with $\mathscr{A} = \mathscr{A}_{\mathscr{L}_{\mathscr{A}}}$, more explicitly

$$\mathscr{A}\mathfrak{F} = \bigcup_{\mathfrak{F}*\mathfrak{G}} \bigcap_{\mathfrak{G}*\mathfrak{H}} \mathscr{A}\mathfrak{H} . \tag{20}$$

As a consequence of continuity from above, one has the equation

$$\mathscr{A} \bigcap_{\mathfrak{G}\in\Gamma} \mathfrak{G} = \bigcup_{\mathfrak{G}\in\Gamma} \mathscr{A}\mathfrak{G} \quad \text{(for any finite } \Gamma \subset \Phi(X)) . \tag{21}$$

Mind that this equation is really weaker, whereas its generalization to completely arbitrary sets $\Gamma \subset \Phi(X)$ would be much stronger than continuity from above. Yet combined with continuity from below, condition (21) (for finite sets Γ) is equivalent with continuity from above. A relation $\mathscr{A}$ at the same time continuous from below and from above might be called *continuous* or *pretopological.*

Coming to speak about the properties of limits as involved in our Theorem, we are re-entering historical ground. Somehow, the situation for $\mathscr{L}$ is not so pretty as it was for $\mathscr{A}$, since for $\mathscr{L}$, there is no continuity from below, only continuity from above as given by

$$\mathscr{L} = \overline{\mathscr{L}} , \tag{22}$$

where by definition

$$\overline{\mathscr{L}}\mathfrak{F} := \bigcap_{\mathfrak{U}\supset\mathfrak{F}} \mathscr{L}\mathfrak{U} = \bigcap_{\substack{\mathfrak{G}\supset\mathfrak{F} \\ 0\notin\mathfrak{G}}} \bigcup_{\substack{\mathfrak{H}\supset\mathfrak{G} \\ 0\notin\mathfrak{H}}} \mathscr{L}\mathfrak{H} \tag{23}$$

(the reader will excuse our use of the same symbol $\overline{}$ both for $\overline{\mathscr{A}}$ and the dual $\overline{\mathscr{L}}$!). Again, the continuous from above limit operators $\mathscr{L} : \Phi(X) \to \mathfrak{P}(X)$ are in one-to-one correspondence with their restrictions $\mathscr{L}|\Omega(X)$, i.e. with all mappings (pseudo-topologies) $\Omega(X) \to \mathfrak{P}(X)$; again, any continuous from above $\mathscr{L}$ is also called *pseudo-topological* as in our Theorem. So one obtains any pair of linked pseudo-topological operators $\mathscr{A}$, $\mathscr{L}$ by extending a quite arbitrary mapping $\Omega(X) \to \mathfrak{P}(X)$ in two evident dual manners; then in particular $\mathscr{A}|\Omega(X) = \mathscr{L}|\Omega(X)$. Let us remark that in general

$$\mathscr{L}, \overline{\mathscr{L}} \subset \mathscr{L}_{\mathscr{A}_{\mathscr{L}}} \subset \mathscr{L}_{\mathbf{C}_{\mathscr{L}}} = \mathscr{L}_{\mathfrak{B}_{\mathscr{L}}} . \tag{24}$$

Our Theorem states that continuity from above, $\mathscr{L} = \overline{\mathscr{L}}$, is equivalent with $\mathscr{L} = \mathscr{L}_{\mathscr{A}_{\mathscr{L}}}$, more explicitly

$$\mathscr{L}\mathfrak{F} = \bigcap_{\mathfrak{F}*\mathfrak{G}} \bigcup_{\mathfrak{G}*\mathfrak{H}} \mathscr{L}\mathfrak{H} . \tag{25}$$

Again, as a consequence of continuity from above, we obtain

$$\mathscr{L} \bigcap_{\mathfrak{G}\in\Gamma} \mathfrak{G} = \bigcap_{\mathfrak{G}\in\Gamma} \mathscr{L}\mathfrak{G} \quad (\text{for any finite } \Gamma \subset \Phi(X)) . \tag{26}$$

Again, this equation is really weaker then continuity from above, whereas its generalization to arbitrary sets Γ,

$$\mathscr{L} \bigcap_{\mathfrak{G}\in\Gamma} \mathfrak{G} = \bigcap_{\mathfrak{G}\in\Gamma} \mathscr{L}\mathfrak{G} \quad (\text{for any } \Gamma \subset \Phi(X)) , \tag{27}$$

is much stronger: in fact, any $\mathscr{L}$ with this stronger condition is defined as *continuous* or – as in our Theorem – *pretopological*. Our Theorem states that this continuity is equivalent with $\mathscr{L} = \mathscr{L}_{\mathbf{C}_{\mathscr{L}}}$, more explicitly

$$\mathscr{L}\mathfrak{F} = \bigcap_{\mathfrak{F}*G} \bigcap_{G*\mathfrak{H}} \mathscr{L}\mathfrak{H} . \tag{28}$$

Let us remark that one may equivalently decompose continuity, i.e. equation (27) (Γ arbitrary), into equation (26) (Γ finite) and the additional equation

$$\mathscr{L} \bigcap_{\mathfrak{G}\in\Gamma} \mathfrak{G} = \bigcap_{\mathfrak{G}\in\Gamma} \mathscr{L}\mathfrak{G} \quad (\text{for any downwards directed } \Gamma \subset \Phi(X)) , \tag{29}$$

such that (29) and (26) constitute a pretopological axiom system for limits somewhat dual to the pretopological axiom system (16) and (21) for $\mathscr{A}$.

Historical comment: Starting from an arbitrary $\mathscr{L}$, $\overline{\mathscr{L}}$ as defined in (23) is nothing but the generalization to arbitrary filters of the afterwards so called star-convergence as introduced for classical sequences (elementary filters) by URYSOHN in order to fill up a gap in the axiom system for limits as given by FRÉCHET (which was the first axiomatic approach to General Topology). In fact, $\mathscr{L} = \overline{\mathscr{L}}$, our continuity from above, is the generalization of URYSOHN's classical axiom as formulated by KURATOWSKI [17] and others; funny to note how one can manage to escape the use of ultrafilters by considering subsequences of subsequences according to identity (23).

On the other hand, within the area of general filters (nets), there are really three different modifications of star-convergence, namely $\overline{\mathscr{L}}$, $\mathscr{L}_{\mathscr{A}_{\mathscr{L}}}$, and $\mathscr{L}_{\mathbf{C}_{\mathscr{L}}} = \mathscr{L}_{\mathfrak{B}_{\mathscr{L}}}$, linked by inclusions (24). Note that $\overline{\mathscr{L}} = \mathscr{L}_{\mathscr{A}_{\mathscr{L}}}$ under some weak monotony assumption on $\mathscr{L}$; but even then $\mathscr{L}_{\mathscr{A}_{\mathscr{L}}}$ and $\mathscr{L}_{\mathbf{C}_{\mathscr{L}}}$ will still differ (this was not the case for classical sequences, cf. e.g. KURATOWSKI [17]!); accordingly, continuity (the pretopological property) is much stronger than continuity from above (the pseudotopological property).

5. **Commutativity of transitions.** The task of proofs of all these statements can be reduced by systematic use of the commutativities in the diagram of our 12 transition mappings between the sets of all $\mathfrak{B}$, $\mathbf{C}$, $\mathscr{A}$, and $\mathscr{L}$ respectively. In fact, from the possible commutativities, all $24 = 4 \times 6$ (4 = number of triangles in the total diagram, 6 = number of possible commutativities in any of these triangles) hold in topological spaces: this was the observation we started from. Now we can prove – as a necessary addition to our Theorem – that all 24 commutativities already hold in our pretopological spaces, i.e. under the pretopological assumptions described above. More generally, most of these commutativities only presuppose much weaker hypotheses or no hypothesis at all. E.g. concerning the triangle ($\mathfrak{B}$, $\mathbf{C}$, $\mathscr{A}$), we have the identities

$$\mathbf{C}_{\mathfrak{B}} = \mathbf{C}_{\mathscr{A}_{\mathfrak{B}}}, \quad \mathscr{A}_{\mathfrak{B}} = \mathscr{A}_{\mathbf{C}_{\mathfrak{B}}}, \quad \mathfrak{B}_{\mathscr{A}} = \mathfrak{B}_{\mathbf{C}_{\mathscr{A}}}, \quad \mathbf{C}_{\mathscr{A}} = \mathbf{C}_{\mathfrak{B}_{\mathscr{A}}}$$

for completely arbitrary $\mathfrak{B}$ and $\mathscr{A}$, for arbitrary $\mathbf{C}$ also the inclusions

$$\mathfrak{B}_{\mathbf{C}} \subset \mathfrak{B}_{\mathscr{A}_{\mathbf{C}}}, \quad \mathscr{A}_{\mathbf{C}} \subset \mathscr{A}_{\mathfrak{B}_{\mathbf{C}}},$$

where equality holds iff $\mathbf{C}$ is monotone. Hence we obtain a pairwise bijective commutative triple correspondence between all monotone $\mathfrak{B}$, all monotone $\mathbf{C}$, and all continuous from below $\mathscr{A}$. Moreover, the pretopological property of one of these three leads to the pretopological property of the corresponding two others. In a similar manner, one discusses triangles ($\mathfrak{B}$, $\mathbf{C}$, $\mathscr{L}$), ($\mathfrak{B}$, $\mathscr{A}$, $\mathscr{L}$), ($\mathbf{C}$, $\mathscr{A}$, $\mathscr{L}$). In the end, we have a pairwise bijective commutative quadruple correspondence between all pretopological $\mathfrak{B}$, $\mathbf{C}$,$\mathscr{A}$, and $\mathscr{L}$, such that a pretopological structure on set X may be established in four different, equivalent languages. We may even define a pretopology on set X as a quadruplet $\tau = (\mathfrak{B}, \mathbf{C}, \mathscr{A}, \mathscr{L})$ where each of the four associated pretopological components determines the other three.

This is the occasion for one additional remark. So far, the points x have been – quite naturally – assumed to be just the elements of carrier X; funny to note that without this assumption, the essence of our considerations remains valid without modifications. The generalization possible by this remark would be for fun only unless there were the so-called δ- or proximity spaces of EFREMOVITCH and SMIRNOV (cf. ČECH [4]), invented as abstract approaches of uniformities in General Topology. These δ-spaces come into our considerations if we now specialize the points x to be the subsets of carrier X instead of its elements. In fact, our binary relation $\mathbf{C}$ then comes to be nothing but the proximity relation $\mathbf{C} \subset \mathfrak{P}(X) \times \mathfrak{P}(X)$ usually denoted δ or p, and our pretopological axioms for $\mathbf{C}$, i.e. that $\mathbf{C}^{-1}x$ is a grillage (grille) for each "point" $x \in X$, is nothing but a part of the usual proximity axioms. Again, we may pass to the operator point of view; again, our pretopological axioms state that operator $\mathbf{C} : \mathfrak{P}(X) \to \mathfrak{P}(\mathfrak{P}(X))$ (associating with each set $B \subset X$ the system $\mathbf{C}B$ of all "points" $x \subset X$ such that $B\,\mathbf{C}\,x$) fulfils the KURATOWSKI axioms $\mathbf{C}(B_1 \cup B_2) = \mathbf{C}B_1 \cup \mathbf{C}B_2$, $\mathbf{C}\emptyset = \emptyset$. But not only this: again, we may pass to the dual relation or operator $\mathfrak{V}$, which again associates with each $x \subset X$ the system $\mathfrak{V}x$ of its proximity neighbourhoods, again we may introduce the associated notions $\mathscr{A}$ and $\mathscr{L}$, thus obtaining a fourfold approach to proximity spaces the detailed discussion of which is left to the reader.

6. The ordered semi-group of pretopologies on set X, additional axioms. The pretopological $\mathfrak{V}$, $\mathbf{C}$, $\mathscr{A}$, and $\mathscr{L}$ considered so far constitute four complete lattices. In more detail, the pretopological $\mathfrak{V}$ and $\mathscr{L}$ even constitute closure systems on the sets of all arbitrary $\mathfrak{V}$ and $\mathscr{L}$ respectively, the pretopological $\mathbf{C}$ a kernel system on the set of all arbitrary $\mathbf{C}$, whereas the system of all pretopological $\mathscr{A}$ – also a complete lattice by inclusion! – is the intersection of the closure system of all continuous from below and the kernel system of all continuus from above (pseudo-topological) $\mathscr{A}$. From the operator point of view, the complete lattice of all pretopological $\mathfrak{V}$, i.e. of arbitrary mappings $\mathfrak{V} : X \to \Phi(X)$, may be simply described as the full direct (cartesian) power $\Phi(X)^X$ of the complete lattice $\Phi(X)$ of all filters on X. According to the fundamental properties of Galois correspondences, the transition mappings (the translations between our four languages) are not only bijective, but also anti-isomorphic or isomorphic with respect to lattice orders (inclusions). In more detail, the comparison between pretopologies $\tau_i = (\mathfrak{V}_i, \mathbf{C}_i, \mathscr{A}_i, \mathscr{L}_i)$ $(i = 1, 2)$ is described by the equivalence

$$\mathfrak{V}_1 \subset \mathfrak{V}_2 \Leftrightarrow \mathbf{C}_1 \supset \mathbf{C}_2 \Leftrightarrow \mathscr{A}_1 \supset \mathscr{A}_2 \Leftrightarrow \mathscr{L}_1 \supset \mathscr{L}_2\,;$$

one says, that τ_1 is coarser than τ_2, $\tau_1 \succ \tau_2$, τ_2 finer than τ_1, $\tau_2 \prec \tau_1$. A *proper pretopology* is a pretopology τ which is coarser than the usual *discrete* (*pre*)*topology* $\tau_d = (\mathfrak{V}_d, \mathbf{C}_d, \mathscr{A}_d, \mathscr{L}_d)$, where

$$\mathfrak{V}_d x = [\![\{x\}, X]\!]\,, \quad \mathbf{C}_d A = A\,,$$

$$\mathscr{A}_d \mathfrak{B} = \bigcap_{A \in \mathfrak{B}} A\,, \quad \mathscr{L}_d \mathfrak{B} = \begin{cases} \{x\} & (\text{if } \mathfrak{B} = \mathfrak{V}_d x)\,, \\ \emptyset & (\text{else})\,, \end{cases}$$

for all $x \in X$, $A \subset X$, $\mathfrak{B} \in \Phi(X)$.

Topologies are usually defined as proper pretopologies which are idempotent, $\mathbf{CC}A = \mathbf{C}A$. More generally, we may establish a semi-group structure in the set of all pretopological (proper or not) $\mathbf{C}$ by the usual composition of operators,

$$(\mathbf{C}_2 \cdot \mathbf{C}_1)\,A = \mathbf{C}_2(\mathbf{C}_1 A)\,.$$

This non-commutative multiplication is not only monotone in both factors, but also distributive with respect to finite suprema (i.e. set-theoretic unions in the relational interpretation) in the lattice of all pretopological $\mathbf{C}$: for all finite T,

$$\left(\bigcup_{t\in T} \mathbf{C}_t\right)\mathbf{C} = \bigcup_{t\in T}(\mathbf{C}_t\mathbf{C}), \quad \mathbf{C}\bigcup_{t\in T}\mathbf{C}_t = \bigcup_{t\in T}(\mathbf{C}\mathbf{C}_t)$$

(the first distributive law even holds for quite arbitrary T). Thus, we obtain a semi-lattice ordered semi-group. The discrete topology $\mathbf{C}_d$ is the unit of this semi-group, moreover – by definition – the unit of the restricted semi-lattice of proper pretopologies.

What remains is to express this multiplication of pretopologies $\tau_i = (\mathfrak{V}_i, \mathbf{C}_i, \mathscr{A}_i, \mathscr{L}_i)$ $(i = 1, 2)$ in the other three languages. We are going to do this for $\mathfrak{V}$. Here we have two possibilities. First, we may consider the interior operators $\mathbf{J} = \mathfrak{V}^{-1}$: $: \mathfrak{P}(X) \to \mathfrak{P}(X)$. By monotony of $\mathbf{C}$, the first formula of $(\mathfrak{V} - \mathbf{C})$ is equivalent with the usual interior formula

$$\mathbf{J}A = \mathfrak{V}^{-1}A = X - \mathbf{C}(X - A).$$

Now in the product pretopology $\tau = (\mathfrak{V}, \mathbf{C}, \mathscr{A}, \mathscr{L})$ of factors τ_i, where $\mathbf{C} = \mathbf{C}_2 \cdot \mathbf{C}_1$, the interior operator $\mathbf{J} = \mathfrak{V}^{-1}$ is nothing but the application of interior operators $\mathbf{J}_2 = \mathfrak{V}_2^{-1}$ after $\mathbf{J}_1 = \mathfrak{V}_1^{-1}$,

$$\mathbf{J}A = \mathbf{J}_2(\mathbf{J}_1A) \quad \text{or} \quad \mathfrak{V}^{-1}A = \mathfrak{V}_2^{-1}(\mathfrak{V}_1^{-1}A).$$

The other description of product neighbourhoods makes use of a double extension procedure for neighbourhoods of a quite arbitrary pretopology τ. This extension procedure follows precisely the two composition laws in the lattice $\Phi(X)$ as described above. First, operator $\mathfrak{V} : X \to \Phi(X)$ is extended from points, i.e. essentially principal ultrafilters, to sets, i.e. arbitrary principal filters, by the – more or less classical – definition

$$\mathfrak{V}A = \bigcap_{x\in A}\mathfrak{V}x$$

(note that principal filter $[\![A, X]\!]$ is the intersection of principal ultrafilters $[\![\{x\}, X]\!]$ of elements $x \in A$). In a second step, this operator $\mathfrak{V} : \mathfrak{P}(X) \to \Phi(X)$ is extended to completely arbitrary filters $\mathfrak{B}$ by the definition

$$\mathfrak{V}\mathfrak{B} = \bigcup_{A\in\mathfrak{B}}\mathfrak{V}A = \bigcup_{A\in\mathfrak{B}}\bigcap_{x\in A}\mathfrak{V}x.$$

So at last, we have a monotone operator $\mathfrak{V} : \Phi(X) \to \Phi(X)$ which assigns to each filter $\mathfrak{B}$ (principal or not) its neighbourhood filter $\mathfrak{V}\mathfrak{B}$: B is a neighbourhood of filter $\mathfrak{B}$ iff it is a neighbourhood of $\mathfrak{B}$-almost all points $x \in X$. (This is essentially the $\mathfrak{B}$-filtered sum of all neighbourhood filters $\mathfrak{V}x$, $x \in X$, as introduced by GRIMEISEN [9] and applied frequently in modern model theory; another description of $\mathfrak{V}\mathfrak{B}$ may be given by means of the algebraic apparatus constructed by KOWALSKY [15].)

It is now easy to see that the product neighbourhoods $\mathfrak{V}$ of factors $\mathfrak{V}_1$, $\mathfrak{V}_2$ can be described by the composition law

$$\mathfrak{V}\mathfrak{B} = \mathfrak{V}_1(\mathfrak{V}_2\mathfrak{B}),$$

for any filter $\mathfrak{B}$, in particular for any point x; so we may define $\mathfrak{V}_1 \cdot \mathfrak{V}_2 := \mathfrak{V}$. Remember that $\mathbf{C} = \mathbf{C}_2 \cdot \mathbf{C}_1$; so the transition from $\mathbf{C}$ to $\mathfrak{V}$ is not only an anti-isomorphism of lattices, but also of semi-groups. The idempotency of proper pretopology, $\mathbf{C}^2 \subset \mathbf{C}$, can now be expressed by $\mathfrak{V}^2 \supset \mathfrak{V}$: this is the last of Hausdorff's classical neighbourhood axioms.

The translation of multiplication of pretopologies into the languages $\mathscr{A}$ and $\mathscr{L}$ can be taken from the work of GRIMEISEN.

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