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ON HOMEOMORPHISMS OF ∞-DIMENSIONAL BUNDLES

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We announce here several generalizations of results in [On homeomorphisms of infinite-dimensional bundles I, II, and III] to not necessarily separable locally trivial fibre bundles $\xi = (E, p, B)$ over polyhedron base space B with fibre a paracompact manifold M modeled on some Fréchet space F homeomorphic to F^∞, the countable infinite product of F by copies of itself. (Starting with Theorem 2 we will let M and (E, p, B) denote respectively such manifolds and bundles.) The following is our main lemma.

Theorem 1 [1]. *Let $\xi = (E, p, B)$ be a fibre bundle over (Hausdorff) space B with fibre F a metric absolute retract. Let $A \subset E$ be a closed set such that for each $b \in B$, the inclusion $j : p^{-1}(b) \setminus (A \cap p^{-1}(b)) \to p^{-1}(b)$ is a homotopical equivalence. Suppose (K, L) is a locally finite simplicial pair and f a map of $|K|$ into B, then each lifting f_1 of $f \mid |L|$ into $E \setminus A$ (that is, $f_1 : |L| \to E \setminus A$ such that $pf_1 = f \mid |L|$) can be extended to a lifting f^* of f into $E \setminus A$.*

A closed subset A of a space X is a *Z-set* if Interior $(A) = \emptyset$ and for each non-empty homotopically trivial open subset U of X, $U \setminus A$ remains homotopically trivial. By virtue of Theorem 1 we prove

Theorem 2. *Let K be a closed set in the total space $M \times B$ of the product bundle $(M \times B, p, B)$ over polyhedron B satisfying that for each $b \in B$, $K \cap p^{-1}(b)$ is a Z-set in $p^{-1}(b)$. Let $\mathcal{U}$ denote any open cover of $M \times B$. Then there is a fibre-preserving (that is, each $p^{-1}(b)$ being mapped into itself by f_t) homotopy $F = \{f_t\}$ of $M \times B$ into itself such that $f_0 =$ identity, $\mathrm{cl}\,(f_1(M \times B)) \cap K = \emptyset$ and F is limited by $\mathcal{U}$ (that is, each $F(\{x\} \times [0, 1]) \subset U$ for some $U \in \mathcal{U}$).*

The case where $B = \{\text{point}\}$ was announced earlier by D. Henderson. (Incidentally, our proof may be different from his.)

Hereafter all maps f of any $A \subset E$ into E will be fibre-preserving maps, that is, $pf(x) = p(x)$ for any $x \in E$. We also let $K_1, K_2, \ldots$ denote closed subsets of E such that for any $b \in B$, $K_i \cap p^{-1}(b)$ is a Z-set in $p^{-1}(b)$.

Theorem 3. *Let f be a homeomorphism of K_1 onto K_2. Then f can be extended to a homeomorphism $\bar{f}$ of E provided that f is homotopic to the identity on K_1.*

Furthermore, if the homotopy is limited by some open cover $\mathscr{U}$ of E, we may choose $\tilde{f}$ to be isotopic to the identity and the isotopy be limited by $\mathrm{St}^{(4)}(\mathscr{U})$.

(We define $\mathrm{St}(\mathscr{U})$ to be the open cover of E consisting of all sets V such that for some $U \in \mathscr{U}$, $V = \bigcup\{W \in \mathscr{U} : W \cap U \neq \emptyset\}$.)

Theorem 4. *E is homeomorphic to* $E \setminus \bigcup_{i \geq 1} K_i$. *Furthermore, if we let φ denote the collection of all such homeomorphisms and if $\mathscr{U}$ is any open cover of E, we may choose $f \in \varphi$ to be isotopic to the identity and the isotopy be limited by $\mathscr{U}$.*

Using the same technique as in [4] we prove

Theorem 5. *Let $(M \times \Delta_n, p, \Delta_n)$ be a product bundle over n-simplex Δ_n and $f : M \times \Delta_n \to M \times \Delta_n$ be a map such that $f \mid M \times \partial\Delta_n$ is a homeomorphism of $M \times \partial\Delta_n$. Then $f \mid M \times \partial\Delta_n$ can be extended to a homeomorphism F of $M \times \Delta_n$. Furthermore, if $n = 1$ and the homotopy $\{f_t = f \mid M \times \{t\}\}$* is *limited by some open cover $\mathscr{U}$ of M, we may choose F to be limited by* $\mathrm{St}^{(10)}(\mathscr{U})$.

Corollary. *Any two homeomorphisms of M are isotopic if and only if they are homotopic.*

(This result was also announced by T. A. Chapman.)

References

[1] *R. Y. T. Wong:* On homeomorphisms of infinite-dimensional bundles, I.
[2] *T. A. Chapman and R. Y. T. Wong:* On homeomorphisms of infinite-dimensional bundles, II.
[3] *T. A. Chapman and R. Y. T. Wong:* On homeomorphisms of infinite-dimensional bundles, III.
[4] *R. Y. T. Wong:* Parametric extensions of homeomorphisms for Hilbert manifolds.