Toposym 3

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DIRECT LIMITS OF HAUSDORFF SPACES

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It is well-known that for topological spaces direct limits have generally very poor preservation properties. Indeed, the direct limit of compact separable metrizable spaces can be an infinite indiscrete space [2, p. 422]. However if all of the bonding maps are injective, the direct limit of T_1 spaces will be T_1. Likewise if all of the bonding maps are closed embeddings, the direct limit of a countable well-ordered spectrum of T_4 spaces will be T_4. (This preservation fails, however, if the countability condition is dropped.) Recently Herrlich [3] has shown that the direct limit of a countable well-ordered spectrum of completely regular Hausdorff spaces with closed embedding bonding maps may fail to be Hausdorff. In this paper we will exhibit sufficient conditions for the Hausdorff property to be preserved under direct limits.

In [1] Banaschewski considered, for any T_0 space, X, a space consisting of all open filters on X. To achieve our direct limit results, we use essentially the same type of construction.

Definition 1. If X is a space, then an *open filter* on X is a nonempty collection F of open sets of X such that:

(i) $\emptyset \notin F$,

(ii) $U, V \in F \Rightarrow U \cap V \in F$, and

(iii) $W \supseteq U \in F \Rightarrow W \in F$.

For any space X, let γX denote the set of all open filters on X. If U is an open set in X, let $U^{\#} = \{F \in \gamma X \mid U \in F\}$. Then $\{U^{\#} \mid U \text{ is open in } X\}$ is a base for a topology on γX. Henceforth γX will denote this space.

Definition 2. If X and Y are spaces and $f\colon X \to Y$ is a continuous function, then

(1) f is said to be *dense* provided that $f[X]$ is dense in Y; and in this case (f, Y) is called a *range* of X.

(2) f is said to be *relatively open* provided that for each open set U of X, $f[U] =$ $= W \cap f[X]$ for some open set W of Y.

(3) f is said to be an *embedding* provided that it is injective and relatively open.

(4) (f, Y) is called an *extension* of X provided that f is a dense embedding.

(5) (f, Y) is called a *C-distinguishable range* of X provided that it is a range of X and for each $y \in Y$ and each closed subset A of Y not containing y, there is an open neighborhood U of y such that $f^{-1}[U] \nsupseteq f^{-1}[V]$ for any open set V of Y that meets A.

The following proposition shows that if X is a T_0-space, then γX is a T_0-compactification of X which can be thought of as a "universal" C-distinguishable extension of X.

Proposition 1. *For every space X, γX is a compact T_0-space with the following properties*:

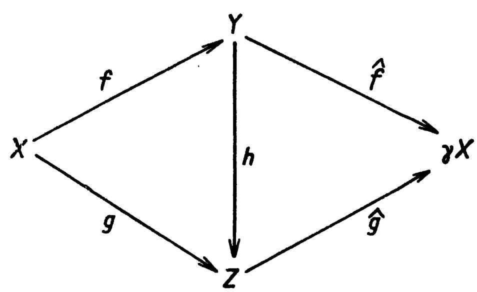
(1) *If (f, Y) is a range of X, then there exists a continuous function $\hat{f}: Y \to \gamma X$ defined by: $\hat{f}(y) =$ the open filter generated by $\{f^{-1}[U] \mid U$ is an open neighborhood of $y\}$.*

(2) *If (f, Y) is a T_0 C-distinguishable range of X, then $\hat{f}$ is an embedding.*

(3) *$\hat{1}_X: X \to \gamma X$ is an embedding if and only if X is a T_0-space.*

(4) *If X is a T_0-space, then the set of all subspaces of γX that contain $\hat{1}_X[X]$ is (up to homeomorphisms) precisely the class of all T_0 C-distinguishable extensions of X.*

(5) *If (f, Y) and (g, Z) are ranges of X and $h: Y \to Z$ is an embedding for which $g = h \circ f$, then $\hat{f} = \hat{g} \circ h$; i.e., if the left-hand triangle of the diagram*



commutes, then so does the right-hand triangle.

Definition 3. Let I be a directed set and let $(X_i, g_{ij})_I$ be a direct spectrum of spaces and bonding maps over I.

(1) $(Y, f_i)_I$ is called a *natural source* for the spectrum provided that:

(i) Y is a space,

(ii) for each $i \in I$, $f_i: Y \to X_i$ is a continuous function, and

(iii) if $i, j \in I$ and $i \leq j$, then $g_{ij} \circ f_i = f_j$.

Dually,

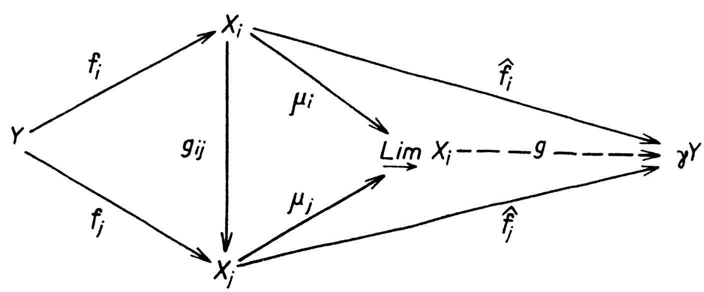
(2) $(f_i, Y)_I$ is a *natural sink* for the spectrum provided that $f_i : X_i \to Y$ and $f_i = f_j \circ g_{ij}$ whenever $i \leqq j$.

(3) $(f_i, Y)_I$ is a *direct limit* of the spectrum provided that it is a maximal natural sink for it.

We will denote the direct limit of the spectrum by $(\mu_i, \varinjlim X_i)_I$.

Theorem 1. *Let $(X_i, g_{ij})_I$ be a direct spectrum of Hausdorff spaces and bonding maps over a directed set I. If each g_{ij} is an embedding and if there exists a natural source $(Y, f_i)_I$ for the spectrum, where Y is any space and each f_i is dense, then $\varinjlim X_i$ is a Hausdorff space.*

Proof. By Proposition 1, for all $i, j \in I$ there exist continuous functions $\hat{f}_i : X_i \to \gamma Y$ and $\hat{f}_j : X_j \to \gamma Y$ such that $\hat{f}_i = \hat{f}_j \circ g_{ij}$. Hence $(\hat{f}_i, \gamma Y)_I$ is a natural sink for the spectrum, so that by the definition of direct limit there is a (unique) continuous function g: $\varinjlim X_i \to \gamma Y$ such that for all $i, j \in I$ the diagram



commutes.

If x and y are distinct points in $\varinjlim X_i$, then, since I is directed, there is some $k \in I$ and points $a, b \in X_k$ such that $\mu_k(a) = x$ and $\mu_k(b) = y$. Since X_k is Hausdorff, there are disjoint open sets U and V such that $a \in U$ and $b \in V$. Then since $\emptyset$ cannot belong to any filter, $(f_k^{-1}[U])^{\#}$ and $(f_k^{-1}[V])^{\#}$ are disjoint open sets in γY; so that $g^{-1}[(f_k^{-1}[U])^{\#}]$ and $g^{-1}[(f_k^{-1}[V])^{\#}]$ are disjoint open neighborhoods of x and y, respectively.

Lemma 1. *Let Y be a Hausdorff space and let $f : X \to Y$ and $h : Y \to Z$ be continuous functions for which f is dense, h is relatively open, and $h \circ f$ is a dense embedding. Then h is an embedding.*

Theorem 2. *Let $(X_i, g_{ij})_I$ be a direct spectrum of Hausdorff spaces and bonding maps over a directed set I. If each g_{ij} is relatively open and if there exists* *a natural source $(Y, f_i)_I$ for the spectrum, where Y is any space and each f_i is a dense embedding, then $\underrightarrow{\mathrm{Lim}}\, X_i$ is a Hausdorff space.*

Proof. By the lemma, each g_{ij} must be an embedding. Apply Theorem 1.

Corollary 1. *If $(X_i, g_{ij})_I$ is a direct spectrum of Hausdorff spaces and dense embedding bonding maps over a directed set I, then $\underrightarrow{\mathrm{Lim}}\, X_i$ is a Hausdorff space.*

Proof. If $I = \emptyset$, then $\underrightarrow{\mathrm{Lim}}\, X_i$ is the empty space which is Hausdorff. If $I \neq \emptyset$, pick $\hat{\imath} \in I$ and let $J = \{i \in I \mid i \geq \hat{\imath}\}$. Then J is cofinal in I; so $\underrightarrow{\mathrm{Lim}}_J\, X_j \cong \underrightarrow{\mathrm{Lim}}_I\, X_i$. But $(X_i, g_{ij})_J$ is a natural source for $(X_i, g_{ij})_J$; so that, by Theorem 2, $\underrightarrow{\mathrm{Lim}}_J\, X_j$ is Hausdorff.

It seems difficult to weaken the hypotheses of the above theorems. Indeed, Dugundji [2, p. 422] has given an example of a direct spectrum of spaces (each homeomorphic to the unit circle) which satisfies all the hypotheses of Theorem 1, except that the bonding maps are not injective. They are, however, relatively open, so that the same example satisfies all of the hypotheses of Theorem 2, except that the natural source maps are not injective. Yet the direct limit space is infinite and indiscrete.

Also Herrlich [3] has given an example of a direct spectrum of completely regular Hausdorff spaces which satisfies all of the hypotheses of Theorems 1 and 2 except that the natural source maps are not dense. (They are, however, embeddings.) But the direct limit space is not Hausdorff.

The following example shows that the hypotheses in the theorems that the bonding maps be relatively open cannot be deleted. Indeed, in the example each space is separable, metrizable and zero-dimensional, and each connecting map is a bijection and is relatively open at all points except one, yet the direct limit space is not Hausdorff.

Example 1. For each positive integer k, let X_k be the rational numbers in the closed interval $[0, 1]$ with the points $\{1/2^n \mid n > k\}$ removed and the points $\{1/2^i \mid 1 \leq \leq i \leq k\}$ identified. For each $n < m$, let g_{nm} be the obvious continuous bijection from X_n to X_m and let $(\mu_n, \underrightarrow{\mathrm{Lim}}\, X_n)$ be the direct limit of the spectrum $(X_n, g_{nm})_N$. Then the points $\mu_1(0)$ and $\mu_1(\frac{1}{2})$ do not have disjoint neighborhoods in $\underrightarrow{\mathrm{Lim}}\, X_n$.

It is also natural to ask whether or not the conclusions of the above theorems can be strengthened; i.e., whether or not stronger separation properties will be preserved by direct limits when the bonding maps are all dense embeddings. The following example shows that stronger separation properties are not preserved since the direct limit of a direct spectrum of separable, metrizable, zero-dimensional spaces with dense embedding bonding maps may fail to be Urysohn; i.e., there may exist distinct points which do not have disjoint closed neighborhoods.

Example 2. For each positive integer k, let X_k be the rational numbers in the closed interval $[-1, 1]$ with the points $\{1 - 1/2^n, -1 + 1/2^n \mid n \geq k\}$ removed, and for $1 < i < k$ each pair of points $(1 - 1/2^i, -1 + 1/2^i)$ identified. For each $n < m$, let g_{nm} be the obvious inclusion map from X_n to X_m. Then $(X_n, g_{nm})_N$ is a direct spectrum with dense embedding bonding maps. However if $(\mu_n, \underrightarrow{\text{Lim}}\, X_n)$ is the direct limit of the spectrum, then $\mu_1(1)$ and $\mu_1(-1)$ do not have disjoint closed neighborhoods in $\underrightarrow{\text{Lim}}\, X_n$.

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