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ON A CONVERGENCE PROPERTY OF SET ALGEBRAS

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Let X be a nomenpty set, 2^X the algebra of all subsets of the set X and λ the *convergence closure operator* on 2^X. We recall its definition. For each $\mathfrak{M} \subset 2^X$,

$$\lambda\mathfrak{M} = \{A;\ A \in 2^X \text{ and there is a sequence of sets } A_n \in \mathfrak{M} \text{ such that}$$

$$A = \lim_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\}\,.$$

A *power* of λ is defined by the transfinite induction: $\lambda^0\mathfrak{M} = \mathfrak{M}$, $\lambda^\alpha\mathfrak{M} = \bigcup_{\beta<\alpha} \lambda(\lambda^\beta\mathfrak{M})$ for an ordinal $\alpha \neq 0$ and $\mathfrak{M} \subset 2^X$.

Let a set algebra $\mathfrak{A}$, $\mathfrak{A} \subset 2^X$, be given. It has been noticed (see [2]) that $\lambda^\alpha\mathfrak{A}$ is also a set algebra for an arbitrary ordinal α and $\lambda^{\omega_1}\mathfrak{A}$ (ω_1 = the first uncountable ordinal) is equal to the σ-algebra generated by $\mathfrak{A}$. An easy completion of the well-known statement (see [1], Chap. 1, Exercise 13) claims:

(1) The image $\mathrm{P}[\lambda^{\omega_1}\mathfrak{A}] = \{\mathrm{P}A;\ A \in \lambda^{\omega_1}\mathfrak{A}\}$ is a closed subset of the real line for each probability measure P.

J. Novák has raised the problem to find the least ordinal α such that $\mathrm{P}[\lambda^\alpha\mathfrak{A}]$ is always closed. The answer is given by

Theorem. *The number $\alpha = 2$ is the least ordinal such that $\mathrm{P}[\lambda^\alpha\mathfrak{A}]$ is closed.*

Proof. 1) $\mathrm{P}[\lambda^2\mathfrak{A}]$ is closed. Let a real number a belong to the closure of $\mathrm{P}[\lambda^2\mathfrak{A}]$. From (1) it follows that there is $B \in \lambda^{\omega_1}\mathfrak{A}$ such that $\mathrm{P}B = a$. The definition of the outer measure implies the existence of sets $B_{ni} \in \mathfrak{A}$, $n = 1, 2, \ldots$, $i = 1, 2, \ldots$, such that $a \leqq \mathrm{P}(\bigcup_{i=1}^{\infty} B_{ni}) \leqq a + 1/n$ and $B \subset \bigcup_{i=1}^{\infty} B_{ni} \in \lambda\mathfrak{A}$ for each $n = 1, 2, \ldots$. Hence $a = \mathrm{P}(\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni}) \in \mathrm{P}[\lambda^2\mathfrak{A}]$.

2) The image $\mathrm{P}[\lambda\mathfrak{A}]$ need not be closed as the following example shows.

Let R denote the real line, $\mathfrak{R}$ the algebra generated by semiclosed intervals of the form $\langle a, b)$, $a, b \in R$.

Lemma. *If $A_n \in \mathfrak{R}$, $n = 1, 2, \ldots$, $A = \lim_{n\to\infty} A_n$, then there is a set Y, $\emptyset \neq Y \in \mathfrak{R}$ such that $Y \subset A$ or $Y \cap A = \emptyset$.*

Proof. We denote $B_i = R + A_i + A_{i+1}$, $C_i = \bigcap_{n=i}^{\infty} B_n$, $i = 1, 2, \ldots$, where $+$ denotes the symmetric difference. Evidently $B_i \in \mathfrak{R}$ and

$$\bigcup_{n=1}^{\infty} C_n = R\,. \tag{2}$$

Now two cases are possible:

1) There is a natural number n_0 such that $C_{n_0} = R$. Then $B_{n_0} = B_{n_0+1} = \ldots = R$ and $A_{n_0} = A_{n_0+1} = \ldots = A$. At least one of the sets $Y = A_{n_0}$ or $Y = R - A_{n_0}$ possesses the declared property.

2) All the sets $C_n \neq R$. Then there is $B_k \neq R$. We choose a compact non-degenerate interval $T_1 \subset R - B_k$. Suppose that compact non-degenerate intervals $T_1 \supset T_2 \supset \ldots \supset T_n$ have been constructed in such a way that $T_i \cap C_i = \emptyset$, $i = 1, 2, \ldots, n$. Denote $T_n^* = T_n - \{r\}$, where r is the right end point of T_n. Now two cases are possible:

a) $T_n^* \subset C_{n+1}$. Then $A_{n+1} \cap T_n^* = A_{n+2} \cap T_n^* = \ldots = A \cap T_n^*$ (otherwise there would be $k > n$ and a point $x \in (A_k + A_{k+1}) \cap T_n^* \subset (R - B_k) \cap C_{n+1} = \emptyset$). Hence the set Y from the Lemma can be found by using a nonempty measurable subset of $T_n^* \cap A$ or $T_n^* - A$.

b) $T_n^* \not\subset C_{n+1}$. Then there is a point $x \in T_n^*$, $x \notin C_{n+1}$, i.e. there is $j \geqq n + 1$ such that $x \notin B_j$. We pick out a non-degenerate compact interval $T_{n+1} \subset T_n^* - B_j$.

In the case a) we have the set Y as desired in the Lemma. If the case a) does not occur, then we have a non-increasing sequence of non-degenerate compact intervals T_n. The intersection of it is disjoint with $\bigcup_{n=1}^{\infty} C_n$ and nonempty. We get a contradiction with (2).

Example. Let $Q = \{q_1, q_2, \ldots\}$ be the set of all rational numbers, $s_n = q_n + \sqrt{2}$, $S = \{s_1, s_2, \ldots\}$. A probability P on $\lambda^2\mathfrak{R}$ is defined by the relations $P(\{q_n\}) = 2/3^{2n-1}$, $P(\{s_n\}) = 2/3^{2n}$, $P(R - Q - S) = 0$. It is easy to see that sets $A_n \in \mathfrak{R}$ can be chosen in such a way that $q_i \in A_n$ and $s_i \notin A_n$ for $i = 1, 2, \ldots, n$. Then $(3/4)(1 - 1/9^n) = \sum_{i=1}^{n} Pq_i \leqq PA_n \leqq 1 - \sum_{i=1}^{n} Ps_i = 3/4 + 1/(4 \,.\, 9^n)$ and hence $\lim_{n\to\infty} PA_n = 3/4$. Now, let A be any set of $\lambda^2\mathfrak{R}$ such that $PA = 3/4$. Then the uniqueness of the ternary expansion $3/4 = \sum_{i=1}^{\infty} 2/3^{2i-1} = \sum_{i=1}^{\infty} Pq_i = P(Q)$ implies $Q \subset A$ and $A \cap S = \emptyset$. Then $A \notin \lambda\mathfrak{R}$ as follows from the Lemma and from the fact that Q and S are dense subsets of R. It follows that $3/4$ is a point of the closure of the P-image of $\mathfrak{R}$ but there is no element $A \in \lambda\mathfrak{R}$ such that $PA = 3/4$.

Remark. Part 1) of the proof of the Theorem can be proved without using an outer measure. The outer measure can be replaced by Marczewski's characteristic function of a sequence of sets (see [3]). Problems and importance of elimination of the notion of an outer measure from measure theory are treated in [2].

References

[1] *M. Loève:* Probability theory. 2nd edition, Van Nostrand, Princeton, 1960.

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