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ON CARDINAL INVARIANTS

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We call cardinal invariants such topological invariants which assume values in the class of all cardinal numbers. These invariants play a very important role in all branches of general topology. The first of them appeared at the earliest stages of development of general topology and showed their importance at once. For example, the second axiom of countability is the basic condition in the classical metrization criteria of Urysohn. The weight of a space X (denoted $w(X)$), the character of a point x in the space X (denoted $\chi(x, X)$) are other examples of classical cardinal invariants. The cardinal-valued topological invariants are very essential in many theorems on classification of topological spaces and continuous mappings. Even the bicompactness property may be considered from this point of view – the definition of bicompactness in terms of complete accumulation points shows it clearly.

The constant and reasonable use of cardinal invariants in different areas of general topology stimulated the appearance of many new interesting invariants of this kind. So it became quite necessary to clarify the interrelations between them – in other words, to classify them, to develop their theory.

Many good results were obtained in this direction in the course of the last five years. I do not intend to give a full survey. This would need a book. In fact, such a book has been written recently by a well-known specialist I. Juhasz. I will present mainly some of my results.

The notation follows that in [3]. In particular, I write $c(X)$ for the Suslin number of the space X, i.e. for the least infinite cardinal τ such that the cardinal of every disjoint family γ of nonempty open subsets of the space X is $\leqq \tau$. I put $cc(X) = \sup \{c(Y) : Y \subset X\}$. All spaces considered are Hausdorff.

1. Here I describe some circumstances in which the condition $c(X) \leqq \aleph_0$ implies paracompactness or even metrizability of X.

1.1. Theorem. *If X is a Čech-complete σ-metacompact and $c(X) = \aleph_0$, then X is Lindelöf.*

(Following F. D. Tall I call a space X σ-metacompact if each open covering of this space has a σ-point finite open refinement. Hence all metacompact and all screenable spaces are σ-metacompact.)

The crucial role in the proof of this theorem is played by the following:

1.2. Lemma. *If X is Čech-complete, $c(X) = \aleph_0$ and γ is a point finite family of open sets in X, then $|\gamma| \leqq \aleph_0$.*

The following result constitutes an essential step in the proof of this lemma.

1.3. Lemma. *If $c(X) = \aleph_0$, k is a natural number and γ is a family of open sets in X such that no point of X belongs to more than k elements of γ, then $|\gamma| \leqq \aleph_0$.*

Notice that we do not suppose here the space X to be Čech-complete.

Besides Theorem 1.1, other interesting conclusions follow from Lemma 1.2. Let us show now some of them.

1.4. Theorem. *If X is metacompact, locally Čech-complete and has locally a countable Suslin number, then X is strongly paracompact.*

1.5. Theorem. *If X is σ-metacompact locally bicompact perfectly normal space, then X is strongly paracompact.*

1.6. Theorem. *If X is a Čech-complete space with a σ-point finite base $\mathscr{B}$ and $c(X) \leqq \aleph_0$, then the base $\mathscr{B}$ is countable (and X is metrizable).*

Pixley and Roy constructed (in an appropriate model of the set theory) an example of a non-metrizable completely regular space X with the uniform base $\mathscr{B}$ (in the sense of P. S. Alexandrov) such that $c(X) = \aleph_0$. As each uniform base $\mathscr{B}$ is σ-point finite it is not possible to extend Theorem 1.6 to all non-Čech complete spaces. (Maybe such an extension is possible in some models.)

Of course, each σ-point finite base is point countable. So in connection with Theorem 1.6 the following question seems to be very interesting.

1.7. Problem. Let X be a Čech-complete space with a point countable base $\mathscr{B}$ and $c(X) \leqq \aleph_0$. Is it true then that the base $\mathscr{B}$ is countable?

2. The remaining part of this article is devoted to an exposition of results on interrelations between cardinal-valued invariants. In the end the reader will find himself completely surrounded by a host of unsolved problems.

I would start with the following example of a result typical for this area.

2.1. Theorem. (A. Hajnal and I. Juhasz, [8]). *If X is first countable and $c(X) = \aleph_0$, then $|X| \leqq 2^{\aleph_0}$.*

An interesting and important generalization of the notion of first countable space is the notion of sequential space. A space X is called sequential if all sequentially closed subsets of X are closed (see [17]). A space X is called Fréchet-Urysohn space (FU-space) if for each $z \in X$ and each $A \subset X$ such that $x \in [A]$ there exists a sequence $\{a_n : n = 1, 2, \ldots\}$ of points belonging to A which converges to x. Evidently, each FU-space is sequential but not conversely. Even a bicompactum exists which is sequential, but not an FU-space [17].

Obviously, each subspace of an FU-space is again an FU-space. Hence each subspace of an FU-space is sequential. But a subspace of a sequential space need not be sequential. It is easy to see that a sequential spece X is an FU-space if and only if each subspace of the space X is sequential. In fact, a stronger result holds: X is an FU-space iff each subspace of X is a k-space [5]. It is worth noticing that the Σ-product of an arbitrary family ξ of real lines (or, more generally, of complete separable metric spaces) is an FU-space ([18]). As the product has a countable Suslin number and the Σ-product is dense in the product, we conclude that the Suslin number of the Σ-product is also countable. So we have exhibited an FU-space X such that $c(X) \leqq \aleph_0$ and $|X| \geqq |\xi|$. And the cardinality of ξ may be as high as we wish. It is easy to see that $\chi(x, X) \geqq |\xi|$ for each $x \in X$. Hence we have proved the following:

2.2. Assertion. *For each cardinal $\tau \geqq \aleph_0$ there exists an FU-space X_τ such that $c(X) \leqq \aleph_0$, $|X| \geqq \tau$ and $\chi(x, X) \geqq \tau$ for all $x \in X$.*

This means that Theorem 2.1 cannot be extended to the class of all Fréchet-Urysohn spaces. But the situation miraculously changes to the best if we concentrate our attention on the class of all bicompact Hausdorff spaces. Of course, for each cardinal τ there exists a bicompactum X such that $|X| \geqq \tau$ and $c(X) = \aleph_0$ – take for example $\mathscr{D}^\tau$ or I^τ (where $I = [0, 1]$ and $\mathscr{D} = \{0, 1\}$). On the other hand, the (Alexandroff's) one point bicompactification A_τ of a discrete space of cardinality $\tau \geqq \aleph_0$ enjoys the following properties: A_τ is first countable at all points but one; A_τ is sequential; A_τ is not homogeneous; $|A_\tau| = \tau$; $c(A_\tau) = \tau$; A_τ is a bicompact Hausdorff space. Results which follow show us that these properties bunched together not by chance.

2.3. Theorem. *If $2^{\aleph_0} = \aleph_1$ and X is a bicompact (Hausdorff) sequential space, then the set of points in which X is first countable is dense in X* [3].

The crucial point in the proof is the following:

2.4. Lemma. *If X is a bicompact sequential space and U is a nonempty open subset in X, then there exists a non-empty closed set $P \subset X$ such that:* (i_1) $P \subset U$; (i_2) $\chi(P, X) \leqq \aleph_0$; (i_3) $|P| \leqq 2^{\aleph_0}$ [3].

Using this lemma and the well known Ramsey type theorem proved by Erdös and Radó, we arrive also to the following result:

2.5. Theorem. *If X is a bicompact sequential space and $c(X) \leqq \aleph_0$, then* $|X| \leqq 2^{\aleph_0}$ [3].

It is useful to look now once more at the spaces A_τ.

2.6. Theorem. *If X is a sequential bicompact homogeneous space, then either* $|X| = 2^{\aleph_0}$ *or* $|X| < \aleph_0$.

Let us sketch the proof of this theorem to reveal the close interrelations between the results mentioned above.

From 2.4 it follows that $\chi(x, X) \leqq 2^{\aleph_0}$ for some $x \in X$. As X is homogeneous, $\chi(x, X) \leqq 2^{\aleph_0}$ for all $x \in X$. In [2] was proved that if X is a bicompact sequential Hausdorff space such that $\chi(x, X) \leqq 2^{\aleph_0}$ for all $x \in X$, then $|X| \leqq 2^{\aleph_0}$. If X has an isolated point, then X is finite. If there are no isolated points in X, then $|X| \geqq 2^{\aleph_0}$ [1]. So in this case $|X| = 2^{\aleph_0}$.

With the aid of GCH (which means: $2^\tau = \tau^+$ for each $\tau \geqq \aleph_0$) we can prove the following

2.7. Theorem. *If X is a homogeneous bicompactum, then $|X|$ is an isolated cardinal number* [3].

Now, we turn to some generalizations naturally connected with the notion of a sequential space.

3.3.1. Let X be a topological space and τ a cardinal number different from zero. For any $A \subset X$ put $[A]_\tau = \bigcup\{[B] : B \subset A \text{ and } |B| \leqq \tau\}$.

It is easy to see that always $[[A]_\tau]_\tau = [A]_\tau$; moreover, the operator $[\ ]_\tau$ is a closure operator on X for some topology on X which will be denoted by $\mathscr{T}_\tau$ provided the given topology on X is denoted by $\mathscr{T}$.

3.2. We say that $A \subset X$ is a G_τ-set (in X) if there is a family γ of open subsets of X such that $\bigcap\{U : U \in \gamma\} = A$ and $|\gamma| \leqq \tau$.

Put $[A]^\tau = \{x \in X\colon$ if $x \in Q$ and Q is a G_τ-set, then $Q \cap A \neq \Lambda\}$. Obviously, $[[A]^\tau]^\tau = [A]^\tau$. The operator $[\ ]^\tau$ is a closure operator for some topology on X which will be denoted by $\mathscr{T}^\tau$. Evidently, $\mathscr{T}_\tau \supset \mathscr{T}$ and $\mathscr{T}^\tau \supset \mathscr{T}$. Usually, $\mathscr{T}^\tau$ and $\mathscr{T}_\tau$ are not comparable and in general, the formula $\mathscr{T}^\tau \cap \mathscr{T}_\tau = \mathscr{T}$ does not hold.

3.3. For example, let $H = (I, \mathscr{T})$ be the space which we obtain by declaring all countable subsets of the segment $I = [0, 1]$, as well as the subsets of I which are closed in the usual topology, to be closed. F. Hausdorff was the first who considered this space. I list some important properties of H: 1) $cc(H) \leqq \aleph_0$; 2) $S(H) > \aleph_0$; 3) $\psi(x, H) = \aleph_0$ for each $x \in H$; 4) $[A]_{\aleph_0} = A$ for each $A \subset H$; 5) $[A]^{\aleph_0} = A$ for each $A \subset H$; 6) $\mathscr{T}_{\aleph_0} = \mathscr{T}^{\aleph_0}$ – discrete topology, hence 7) $\mathscr{T}_{\aleph_0} \cap \mathscr{T}^{\aleph_0} \neq \mathscr{T}$ (because H is not discrete).

3.4. Theorem. [4] *If X is bicompact, then for each $A \subset X$ and each $\tau > 0$*

$$[[A]_\tau]^\tau = [A].$$

I think that this is really an important formula. The proof of this formula, if not long, is not trivial. I wish also to underline that it is extremely general. To derive some valuable consequences from the formula we need two definitions.

3.5. Let X be a space. The least cardinal number τ such that $[A]_\tau = [A]$ for each $A \subset X$ is called the tightness of X and is denoted by $t(X)$. Of course, $t(X) \leqq \chi(X) \leqq w(X)$ for each X. If X is sequential, $t(X) \leqq \aleph_0$. But the converse to the last assertion is not true: If X is countable, then clearly $t(X) \leqq \aleph_0$. However, it is easy to construct a countable space X which is not sequential: take for X the set $N \cup \xi \subset \beta N$, where $\xi \in \beta N \setminus N$ and $N \cup \xi$ is considered as a subspace of βN (βN is the Stone-Čech bicompactification of a countable infinite discrete space N).

3.6. Let X be a space and τ a cardinal number. Suppose that for each ordinal $\alpha < \tau$ a point $x_\alpha \in X$ is chosen. Then we say that $\xi = \{x_\alpha : \alpha < \tau\}$ is a free sequence of length τ if for each $\beta < \tau$ the following condition holds:

$$[\{x_\alpha : \alpha < \beta\}] \cap [\{x_\alpha : \beta \leqq \alpha\}] = \Lambda.$$

Obviously, points of a free sequence in X constitute a discrete subspace of X. Of course this subspace may not be closed. If a closed discrete subspace Y of X is given, any minimal well ordering on Y makes Y a free sequence. Obviously, not each discrete subspace of X can be represented as the set of all points of a free sequence. But each countable discrete subspace is the set of all points of some free sequence.

3.7. Theorem. *Let X be a bicompactum. Then $t(X) = \sup\{\tau :$ there is a free sequence of the length τ in $X\}$.*

The proof of Theorem 3.7 heavily depends on Theorem 3.4.

3.8. Corollary. *If X is a bicompactum, then $t(x) \leqq cc(X)$.*

B. Shapirovskij – a student of mine – was the first to formulate and prove this last assertion. A few days later, knowing the Shapirovskij's result but not its proof I gave an independent proof of 3.8. Working on this proof I found Theorems 3.4 and 3.7. The assertion 3.8 may be easily generalized in the following way:

3.9. Corollary. *If X is a k-space, then $t(x) \leqq cc(X)$.*

Besides the results mentioned above and the following theorem almost nothing is known about the tightness – even in the case of bicompact Hausdorff spaces. Hence more attention should be paid to the following positive fact.

3.10. Theorem. *If X is a dyadic bicompactum, then $t(X) = w(X)$.*

The proof of this assertion, given in [19], makes use of the notion of Dante's space (see [19]) and is based essentially on the Hewitt-Marczewski-Pondiszeri's theorem.

3.11. Corollary. *If X is a dyadic bicompactum, then $cc(X) = w(X)$.*

To prove it, we simply combine 3.8 and 3.10. The original Efimov's proof of 3.11 is rather complicated (see [7]).

It is remarkable that Theorem 3.10 includes also the following well known assertions:

3.12. Corollary (A. S. Esenin-Vol'pin). *If X is a dyadic bicompactum, then $\chi(X) = w(X)$.*

3.13. Corollary (A. V. Arhangel'skij). *If a dyadic bicompactum X is a factor-space of some metric space, then X is metrizable.*

Another series of corollaries of 3.8 holds in a special model of the set theory.

4. There exists (see [14], [10]) a model of set theory in which the following assertion is true:

4.1. (TAM) If X is a space such that $c(X) = \aleph_0$, then for each uncountable family ξ of non-empty open subsets of X there exists an uncountable subfamily $\eta \subset \xi$ which is centered.

4.2. Theorem. *If* (TAM) *holds and X is a bicompactum such that $cc(X) \leqq \aleph_0$, then each subspace of X is separable.*

In other words, if (TAM) holds then bicompact Hausdorff space is hereditary separable if and only if each discrete subspace of this space is countable. The proof of the fact is non-trivial. We essentially use the relation $t(X) \leqq cc(X) \leqq \aleph_0$, which holds by 3.8 (see [5]).

4.3. I. Juhasz proved that if (TAM) holds and X is a first countable bicompact Hausdorff space such that $c(X) = \aleph_0$, then X is separable [10].

There exist such models of the set theory in which not only (TAM) is true but the following relations also hold: $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. In any of these models the following theorem may be proved:

4.4. Theorem. *If X is a sequential bicompactum such that $c(X) = \aleph_0$, then X is separable.*

In the proof, Lemma 2.4 plays a very important role. D. Kurepa [11] proved that the product X of an arbitrary family $\{X_\alpha : \alpha \in A\}$ of spaces such that $c(X) = \tau \geqq \aleph_0$ for each A has a Suslin number not greater than 2^τ (see also [9]). In his other very elegant work [20] D. Kurepa proved that if X is a topological space, its topology being induced by a linear order on X, and $c(X \times X) = \aleph_0$, then $s(X) \leqq \aleph_0$.

Now from the arguments in [15] (see also [12]), it is clear that the following very remarkable theorem holds:

4.5. Theorem. *If* (TAM) *is true,* $X = \prod\{X_\alpha : \alpha \in A\}$ *and* $c(X_\alpha) = \aleph_0$ *for each* $\alpha \in A$, *then* $c(X) = \aleph_0$.

The technique developed to prove the results of the last group yields some conclusions of absolute character – which are true in all reasonable models.

We shall use now the following notation: $ss(X) = \sup\{s(Y) : Y \subset X\}$. Here $s(Y)$ stands for the density of Y.

4.6. Theorem. *Let X be a bicompactum, τ a cardinal number and, for each closed subspace Y of the space X, $s(Y) \leqq \tau$. Then $ss(X) \leqq \tau$.*

We sketch here the proof of 4.6. From the assumptions about X it follows that $cc(X) \leqq \aleph_0$. By 3.8 we have then $t(X) \leqq \aleph_0$. It is sufficient now to use the following simple

4.7. Lemma. *If X is a space, Y a subspace of X, τ a cardinal number and* (a) $[Y] = X$; (b) $t(X) \leqq \tau$; (c) $s(X) \leqq \tau$; *then* $s(Y) \leqq \tau$.

Although the following part is devoted mainly to problems, it involves also a discussion of problems and in its course we also mention some new interesting results.

5.5.1. Problem. Is there a bicompactum X such that $|X| \leqq 2^{\aleph_0}$, $c(X) = \aleph_0$ and $s(X) > \aleph_0$? It is easy to prove that if (TAM) holds and $|X| = \aleph_1$, $c(X) = \aleph_0$, then $s(X) \leqq \aleph_0$. But as far as I know it is not known whether (TAM) is consistent with $2^{\aleph_0} = \aleph_1$. A. Hajnal and I. Juhasz have proved recently that if (TAM) holds and X is a bicompactum such that π-weight of X is less than or equal to $\aleph_1$ and $c(X) = \aleph_0$, then X is separable [13].

5.2. Problem. Let X be a Lindelöf space in which each point is a G_δ-set. Is it true then that $|X| \leqq 2^{\aleph_0}$?

In [2] I showed that the answer is yes if X is first countable Lindelöf space. Now I have the following result:

5.3. Theorem. *If X is a regular Lindelöf space such that $\{x\}$ is a G_δ-set for each $x \in X$ and $t(X) \leqq \aleph_0$, then $|X| \leqq 2^{\aleph_0}$.*

Is it possible to extend this result from regular to Hausdorff spaces?

Problem 5.2 would be settled in the positive way if we could prove one of the following two hypotheses:

5.4. Hypothesis. If X is a Hausdorff space and each point of X is a G_δ-set in X, then there exists a first countable Hausdorff space Y which is a one-to-one continuous image of the space X.

5.5. Hypothesis. If X is a regular Lindelöf space and each point of X is a G_δ-set in X, then there exists a first countable Hausdorff space Y which is a one-to-one continuous image of the space X.

5.7. Problem (A. Hajnal and I. Juhasz). Let X be a bicompactum such that $ss(X) \leqq \aleph_0$. Is it true then that $|X| \leqq 2^{\aleph_0}$?

5.8. Problem (A. Hajnal and I. Juhasz). The same question as in 5.7 for an arbitrary Hausdorff space X.

5.9. Problem (B.A. Efimov). Is it true that for each infinite bicompact Hausdorff space X one of the following two alternatives holds:

1. X contains a non-trivial convergent sequence of points;
2. X contains a topological copy of βN?

Problems 5.7 and 5.9 are closely connected with the following group of problems.

5.10. Problems. Let X be a bicompactum such that $t(X) \leqq \aleph_0$. Which of the following assertions 5.10.1 – 5.10.5 are then true?

5.10.1. X is sequential.

5.10.2. If $2^{\aleph_0} = \aleph_1$, then X is first countable at some point. (See 2.3.)

5.10.3. If $c(X) = \aleph_0$, then $|X| \leqq 2^{\aleph_0}$.

Notice that if $ss(X) \leqq \aleph_0$, then $t(X) \leqq \aleph_0$ and $c(X) = \aleph_0$. Look now at 5.7. See also 2.5.

5.10.4. If $|X| \geqq \aleph_0$, then X contains a non-trivial convergent sequence of points.

5.10.5. If (TAM) and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ holds, and $c(X) = \aleph_0$, then X is separable. (See 4.4.)

5.11. Problem. Let X be a bicompactum and $cc(X) \leqq \aleph_0$. Is it true then that $s(X) \leqq \aleph_0$? (See 4.2.)

5.12. Problem. Is it true that if for each closed subspace Y of a completely regular space X, $s(Y) \leqq \tau$, then $s(Z) \leqq \tau$ for each subspace Z of the space X? (See 4.6 and 3.3.)

5.13. Problem. Let X be a completely regular space such that $cc(X) \leqq \tau$. Is there a bicompact Hausdorff extension bX of X such that $cc(bX) \leqq \tau$?

5.14. Problem. The same question as in 5.13 for X such that $s(Y) \leqq \tau$ for each closed subspace Y of the space X. (We do not seek for a stronger conclusion.)

5.15. Problem. Which spaces of countable tightness can be realized as subspaces of a sequential spaces? (Clearly each subspace of a sequential space has countable tightness; more generally, $t(Y) \leqq t(X)$ as soon as Y is a subspace of X. Let us remark that not each countable space is a subspace of a sequential space.)

5.16. Problem. Which spaces are homeomorphic to subspaces of bicompact Hausdorff spaces of countable tightness, of sequential bicompact Hausdorff spaces, respectively?

It is true that if X is a bicompactum such that $c(X) \leqq \aleph_0$ and $t(X) \leqq \aleph_0$, then $|X| \leqq 2^{2^{\aleph_0}}$ (look now once more at 5.10.3). Hence not every FU-space has a bicompactification with countable tightness (for the proof take an appropriate Σ-product — see the arguments on page 39).

5.17. Problem (F. D. Tall). Suppose (TAM) holds and let X be a Čech-complete first countable space such that $c(X) = \aleph_0$. Is it true then that X is separable? (See 4.3 and 1.7.)

5.18. Problem. Is the product of countably many spaces of countable tightness again a space of countable tightness? V. Malychin proved that the answer is yes if all the factors are bicompact.

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