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ON CONDENSATION NUMBERS

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In this note we consider the weight (=character) of a topological space at a point and some other cardinal numbers which describe the "condensation" of points at a given point.

Notation. The cardinal number of a set B is denoted by card B. We denote by k a fixed sufficiently large cardinal, K denotes the set of cardinals $\leqq k$, and $\hat{K}$ denotes the Kurepa completion of K; l will denote an element of $\hat{K}$. The letter E denotes a topological space, α a point of E, V a neighbourhood of α in E, B a subset of E, $\bar{B}$ the closure of B. The weight of E at α is denoted by $w(\alpha)$.

$P(\alpha, l)$ will be an abbreviation for "if $\alpha \notin B$ and card $B \leqq l$, then $\alpha \notin \bar{B}$". If α is given, $l(\alpha)$ will denote any l for which $P(\alpha, l)$ holds. We can consider the following elements of $\hat{K}$: $\sup l(\alpha)$, $\inf_{\alpha\in E} \sup l(\alpha)$, $\sup_{\alpha\in E} \sup l(\alpha)$. In particular, if $\sup l(\alpha)$ is constant for $\alpha \in E$, then E is "homogeneous" in a certain sense. If not, then the equality $\sup l(\alpha_1) = \sup l(\alpha_2)$ defines a partition of E which may be worth consideration.

Clearly, if $0 \leqq l \leqq \inf_{\alpha\in E} \sup l(\alpha)$, then $P(\alpha, l)$ holds for all $\alpha \in E$.

Proposition. *For every non-isolated* $\alpha \in E$, $w(\alpha) > \sup l(\alpha)$.

This follows at once from the Axiom of Choice; the inequality is strict, due to considering $\hat{K}$ instead of K.

Let $(D, \geqq)$ be a directed set and let $A \subset D$. Let $\mathscr{T}$ denote the collection of all maximal totally ordered subsets T of D such that for any $a \in A$ there is an element $d \in T$ with $a \geqq d$. In the following we assume that A is such that $\bigcup \mathscr{T} = D$.

For every $T \in \mathscr{T}$ let μ_T denote the least cardinality of a cofinal subset of T. The set of all numbers μ_T will be denoted by $M(D, A)$. It can be shown that the supremum (in $\hat{K}$) of the set $M(D, A)$ is a dimension in the sense of [3], [4]. This supremum will be denoted by dep (D, A) and called the depth of D with respect to A; the least cardinal $\geqq$ dep (D, A) will be denoted by dep* (D, A). The least cardinality of a cofinal subset of D will be called the weight of D and denoted by $w(D)$, and the cardinality of $\mathscr{T}$ will be denoted by br (D, A) and called the breadth of D with respect to A.

Clearly, $w(D) \leqq \text{dep}^* (D, A) \text{ br} (D, A)$. Hence, $\text{br} (D, A) = w(D)$ whenever $\text{dep}^* (D, A) < w(D)$.

We denote by D_α the collection of all neighbourhoods of α (in E) ordered by inclusion; A_α will denote the collection of all neighbourhoods of α of the form $E - (x)$, $x \in E$. We shall call dep (D_α, A_α) the depth of E at α.

Problem. Does there exist, for any non-void set M of infinite cardinal numbers, a normal space E and a point $\alpha \in E$ such that $M(D_\alpha, A_\alpha) = M$?

Proposition. *If card B is less than the depth of E at α, then $\alpha \notin \bar{B} - B$. In particular, if* dep* $(D_\alpha, A_\alpha) = w(\alpha)$, *then $\alpha \in \bar{B} - B$ implies $w(\alpha) \leqq$ card B.*

Remarks. 1) The condition dep* $(D_\alpha, A_\alpha) = w(\alpha)$ is more general than the assumption of the existence of a totally ordered basis of neighbourhoods of α. — 2) Using the cardinal $\inf_{\alpha \in E} \sup l(\alpha)$ we obtain a classification of topological spaces which starts with the T_1-spaces (namely, E is a T_1-space iff $\inf_{\alpha \in E} \sup l(\alpha) \geqq 1$). — 3) $\inf_{\alpha \in E} \sup l(\alpha)$ and $\sup_{\alpha \in E} \sup l(\alpha)$ are "dimension functions" in the sense of [3]. — 4) Analogous questions can be investigated in generalized topological spaces; cf. also [1]. — 5) There are certain relations between notions introduced in this note, and U-sefs considered in [2]. — 6) In the author's opinion, the term "condensation number at α" is preferable to "weight".

References

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