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ON A MECHANISM OF CHOOSING MORPHISMS IN CONCRETE CATEGORIES[1])

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It was observed (see, e.g., [2], [3], [4], [5]) that many concrete categories (i.e. categories with fixed forgetful functors) may be viewed upon as follows: a functor $F : \mathbf{Set} \to \mathbf{Set}$ is given, the objects are some couples (X, r) where X is a set and r a subset of $F(X)$, and the morphisms from (X, r) into (Y, s) are those mappings for which $F(f)(r) \subset s$ (or $F(f)(s) \subset r$, if F is contravariant). Thus, e.g., the category of topological spaces and their continuous mappings may be obtained using the contravariant power set functor P^- for F, the category of uniform spaces and uniformly continuous mappings may be obtained using the functor $P^- \circ Q$, where Q sends X into $X \times X$, etc.

In other words, it is often the case that a concrete category $(\mathfrak{K}, U)$ is realizable in an $S(F)$ (to recall the definitions from [2] and [4]: $S(F)$ is the category of all (X, r) with X sets and $r \subset F(X)$ for objects, morphisms from (X, r) into (Y, s) are triples $((X, r), f, (Y, s))$ with mappings $f : X \to Y$ satisfying $F(f)(r) \subset s$ or $F(f)(s) \subset r$ according to the variance of F; $S(F)$ is considered as a concrete category endowed by the forgetful functor sending (X, r) to X and $((X, r), f, (Y, s))$ to f. A concrete category $(\mathfrak{K}, U)$ is said to be realizable in a concrete category $(\mathfrak{L}, V)$ if there is a full embedding $\Phi : \mathfrak{K} \to \mathfrak{L}$ with $V \circ \Phi = U$). The aim of this note is to present a necessary and sufficient condition for realizability in an $S(F)$. The proofs will be outlined very roughly. In detail, they will appear in a longer forthcoming paper.

Obviously, the following two conditions on $(\mathfrak{K}, U)$ are necessary for realizability in an $S(F)$:

(J) If $\alpha : a \to b$ is an isomorphism in $\mathfrak{K}$ and if $U(\alpha) = \mathrm{id}_{U(a)}$, then $\alpha = \mathrm{id}_a$.
(S) For every set X, the class $\{a \mid U(a) = X\}$ is a set.

On the other hand, it is also very easy to show on an ad hoc example that they are not sufficient. We have, however, the following

Theorem 1. *If* $(\mathfrak{K}, U)$ *satisfies* (J), (S) *and*

(R) *for every morphism* α *there are morphisms* β *and* γ *such that* $U(\beta)$ *is one-to-one,* $U(\gamma)$ *onto and* $\alpha = \beta \circ \gamma$,
then there is a covariant F *such that* $(\mathfrak{K}, U)$ *is realizable in* $S(F)$.

[1]) Preliminary communication.

To prove this, we show first that if $(\mathfrak{K}, U)$ satisfies (J), (S) and (R), it is realizable in an $(\mathfrak{L}, V)$ satisfying (J), (S) and

(P) for every morphism α and every two mappings f, g such that $U(\alpha) = f \circ g$ there are morphisms β and γ with $\alpha = \beta \circ \gamma$, $U(\beta) = f$ and $U(\gamma) = g$.

Then, we show that if $(\mathfrak{K}, U)$ satisfies (J), (S) and (P), it is realizable in an $S(F)$ with F constructed as follows:

$$F(X) \quad = \{A \mid A \subset \{a \mid U(a) = X\} \& ((a \in A \& \exists\alpha : a' \to a, U(\alpha) = \mathrm{id}) \Rightarrow a' \in A)\},$$
$$F(f)(A) = \{b \mid U(b) = Y, \exists a \in A, \exists\varphi : b \to a, f = U(\varphi)\} \text{ for } f : X \to Y.$$

Of course, the condition (R) is not necessary. To be able to formulate the necessary and sufficient condition mentioned above, let us now give the following

Definition. Let $(\mathfrak{K}, U)$ be a concrete category, a an object of $\mathfrak{K}$ and $m : X \to U(a)$ a one-to-one mapping. Denote by $\mathscr{S}(m)$ the class of all mappings $f: Y \to X$ such that there is a morphism $\alpha : b \to a$ in $\mathfrak{K}$ with $U(\alpha) = m \circ f$. If $\alpha : a \to b$ is a morphism of $\mathfrak{K}$, a U-image of α is any $\mathscr{S}(m)$ such that there is an onto mapping p with $U(\alpha) = m \circ p$. Two morphisms are said to be equivalent if they have a common U-image. We say that $(\mathfrak{K}, U)$ satisfies (E), if

(E) There is a class M of morphisms of $\mathfrak{K}$ such that

(1) for every morphism α there is a $\beta \in M$ equivalent to α,
(2) for every cardinal $\mathfrak{a}$, $\{\alpha \mid \alpha \in M, \text{card range } \alpha \leqq \mathfrak{a}\}$ is a set.

It is not difficult to see that every $S(F)$ has (E) (one can take the class of all embeddings of naturally induced subobjects into objects $(\mathfrak{a}, r)$ for M), and that full concrete subcategories of concrete categories inherit (E). Thus, we have

Statement. (E) *is a necessary condition for* $(\mathfrak{K}, U)$ *to be realizable in an* $S(F)$.

Further, the following lemma holds

Lemma. *If* $(\mathfrak{K}, U)$ *has* (J) *and* (E), *it is realizable in an* $(\mathfrak{L}, V)$ *with* (J), (R) *and* (S).

Consequently, we obtain

Theorem 2. $(\mathfrak{K}, U)$ *is realizable in an* $S(F)$ *iff it has the properies* (J) *and* (E).

Of the two theorems, of course, Theorem 1 is more applicable, since the properties there are very easy to check. As a corollary we obtain, e.g., that every category of topological spaces in which the morphisms are such that every homeomorphism is an isomorphism, and which satisfies (R) is realizable in an $S(F)$. More generally, this holds for any category of structures in the sense of Bourbaki ([1]) satisfying (R).

References

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