ON INTERNAL CHARACTERIZATIONS OF COMPLETE REGULARITY AND WALLMAN-TYPE COMPACTIFICATIONS

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To give an internal characterization of Tychonoff spaces, O. Frink [2] generalized the method introduced by Wallman [7] to provide Hausdorff compactifications for Tychonoff spaces. His procedure uses a *normal base* of closed sets instead of the family of all closed sets employed by Wallman. A base for the closed sets is a *normal base* if it is closed under the operations of finite unions and finite intersections, and satisfies the following conditions:

(i) for any element H of the base and any $x \in X \setminus H$ there are two elements H_1, H_2 of the base such that

$$H_1 \cup H_2 = X, \quad x \notin H_2, \quad H_1 \cap H = \emptyset,$$

(ii) for any two disjoint elements H_1, H_2 of the base there are two elements H', H'' of the base such that

$$H' \cup H'' = X, \quad H_1 \subset X \setminus H', \quad H_2 \subset X \setminus H''.$$

Frink raised the following questions:

Let X be a compact Hausdorff space and Y a dense subset of X; is there any normal base $\mathfrak{B}$ of the closed sets of Y such that the Wallman-type compactification of Y, $\omega(Y, \mathfrak{B})$ is homeomorphic to X?

He also asked whether $\mathfrak{B}$ can be chosen such that every element of $\mathfrak{B}$ is a zero-set. Such compactifications will be called *z-compactifications.*

E. F. Steiner [6] proved that if there is a normal base of closed sets of a compact space X such that every element of the base is a *regular closed set* then X is a Wallman-type compactification of each of its dense subsets.

In this case, we shall say that X is a *regular Wallman compactification* of each of its dense subsets. He also proved that every compact subspace of the real numbers, or every product of compact subsets of real numbers is a regular Wallman compactification of each of its dense subspaces.

Theorem 1. ([4]) *Every (totally) orderable compact space and even every product of orderable compact spaces is regular Wallman-type and, moreover, a z-compactification of each of its dense subsets.*

J. de Groot and J. M. Aarts [1] gave another internal characterization of complete regularity which is a generalization of Frink's theorem and naturally fits between regularity and normality.

To generalize this, we introduce the following notions:

Definition 1. Two subsets A and B of a space X are said to *be screened by a finite family* $\mathfrak{B}$ if $\mathfrak{B}$ covers X and each element of $\mathfrak{B}$ meets at most one of the sets A and B.

We shall say that two subsets A and B of a space X *are screened by the closed (sub) base* $\mathfrak{T}$ of X if A and B are screened by a finite subcollection of $\mathfrak{T}$.

Definition 2. Two subsets A and B of X are said to *be weakly screened by* $\mathfrak{T}$ if there are $A_i \in \mathfrak{T}$, $i = 1, \dots, n$ and $B_j \in \mathfrak{T}$, $j = 1, \dots, m$ such that

$$A \subset \bigcup_{i=1}^{n} A_i, \quad B \subset \bigcup_{j=1}^{m} B_j$$

and for every $i = 1, \dots, n$, $j = 1, \dots, m$ the subsets A_i and B_j are screened by a finite subcollection of $\mathfrak{T}$.

Theorem 2. *A space X is completely regular if and only if there is a subbase $\mathfrak{T}$ for the closed subsets of X such that:*

(1) (*Weak subbase-regularity.*) *If $S \in \mathfrak{T}$, $x \notin S$, then S and $\{x\}$ are weakly screened by $\mathfrak{T}$.*

(2) (*Weak subbase-normality.*) *Every two disjoint elements of $\mathfrak{T}$ are weakly screened by $\mathfrak{T}$.*

The proofs of these theorems can be found in [4] and [5].

References

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