Melvin Henriksen

On difficulties in embedding lattice-ordered integral domains in lattice-ordered fields

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 183--185.

Persistent URL: http://dml.cz/dmlcz/700738

ON DIFFICULTIES IN EMBEDDING LATTICE-ORDERED INTEGRAL DOMAINS IN LATTICE-ORDERED FIELDS

M. HENRIKSEN

Claremont

A *lattice ordered ring* (or *l-ring*) $A = A(+, \cdot, \vee, \wedge)$ is an abstract algebra closed under four binary operations $+, \cdot, \vee, \wedge$ such that $A(+, \cdot)$ is a ring, $A(\vee, \wedge)$ is a lattice, and if 0 is the identity element of $A(+)$, then

$$a, b \geqq 0 \quad \text{imply that} \quad a + b \geqq 0 \quad \text{and} \quad a \cdot b \geqq 0 .$$

As usual, we say that $a \geqq 0$ if $a \vee 0 = a$, and $a \geqq b$ if $(a - b) \geqq 0$. Moreover, we let $|a| = a \vee (-a)$.

Totally ordered rings and lattice-ordered rings of real-valued functions have long been studied, but the first systematic attack on the structure of l-rings as abstract algebras is due to Birkhoff and Pierce [2]. While they made progress with finite-dimensional l-algebras over totally ordered fields, they got a reasonable structure theory only for subdirect sums of totally ordered rings (which they call *f-rings*). For information about f-rings, see the paper by D. G. Johnson [6], the book by L. Fuchs [5], and for the most thorough summary of the literature on l-rings, see the doctoral dissertation of S. A. Steinberg [7].

The kernel of a homomorphism of one l-ring into another is called an *l-ideal*; I is an l-ideal if it is a ring ideal and $|b| \leqq |a|$, and a in I imply that b is in I. An l-ideal P is said to be *prime* if $|a|A|b|$ contained in P implies a or b is in P. If we let $M(A)$ denote the intersection of all the prime l-ideals of A, then J. E. Diem shows in [4] that $M(A)$ contains all the positive nilpotent elements of A and that $A/M(A)$ is a subdirect sum of l-rings such that

$$a > 0 \quad \text{and} \quad b > 0 \quad \text{imply that} \quad a \cdot b > 0 .$$

He calls such an l-ring by the name *l-domain* and notes that it may have proper divisors of zero, even if we assume also that $a^2 \geqq 0$ for all a in A.

This motivates us to seek answers to the following questions:

(1) *Is it always possible to embedd a (commutative) lattice-ordered integral domain A in a lattice-ordered field (preserving the lattice-order of A).*

By capitalizing on work of Conrad and Dauns in [3], Steinberg exhibits a way of embedding a lattice-ordered integral domain A in lattice-ordered field of formal power series under severe restrictions on the structure of the lattice-ordered group $A(+, \vee, \wedge)$. The general question remains open even if one assumes that $a^2 \geqq 0$ for every a in A.

In [3], Conrad and Dauns ask:

(2) *Under what conditions can the field $Q(A)$ of formal quotients of a lattice-ordered integral domain A be made into a lattice-ordered ring (preserving the lattice-order of A).*

To see that there is a difference between these two questions, we give a non-trivial example.

The ring $R[x]$ of real polynomials becomes an l-ring if we let

$$\sum_{k=1}^{n} a_k x^k \geqq 0 \quad \text{providing} \quad a_k \geqq 0 \quad \text{for} \quad k = 0, 1, \ldots, n\,.$$

Now, $R[x]$ is a sub-l-ring of the lattice-ordered field $R((x))$ of formal Laurent series

$$\sum_{k=-m}^{\infty} a_k x^k$$

ordered by saying that such a series is non-negative if each of its coefficients is non-negative.

But the field of quotients of elements of $R[x]$ is not a sublattice of $R((x))$.

For if α is any real number, then

$$\sum_{k=0}^{\infty} (\cos k\alpha)\, x^k = \frac{1}{2}\left[\frac{1}{x - e^{i\alpha}} + \frac{1}{x - e^{-i\alpha}}\right]$$

is in $R[x]$, while it can be shown that its absolute value $\sum_{k=0}^{\infty} |\cos k\alpha|\, x^k$ does not represent a rational function unless α is a rational multiple of π. (This result may be derived from a general result in [1], but a direct computational proof was obtained by my colleague S. Busenberg.)

Note that this latter result depends essentially on the fact that infinitely many coefficients of the power series in question are irrational, so it seems natural to ask:

(3) *If Q is the field of rational numbers, is the field $Q(x)$ of rational functions, ordered as above, a sublattice of the field $Q((x))$ of formal Laurent series? This is, if*

$$\sum_{k=-m}^{\infty} a_k x^k$$

represents a rational function (in some neighborhood of the origin) where each a_k

is a rational number, does

$$\sum_{k=-m}^{\infty} |a_k| \, x^k$$

represent a rational function? Indeed, what if each a_k is an integer?

A similar question is posed by Steinberg in [7].

I have one small contribution to make towards an answer to (1). Note that the embedding given above of $R[x]$ into $R((x))$ is *order-convex* in the sense that if $|b| \leqq |a|$ and b is in $R[x]$, so is a.

I can show that:

Suppose A is a lattice-ordered integral domain such that $a^2 \geqq 0$ for every a in A, and that A has no nonzero proper l-ideals. Then, if A is an order-convex sub-l-ring of a lattice-ordered field F, then $A = F$.

If we order $R[x]$ by letting

$$\sum_{k=0}^{n} a_k x^k > 0 \quad \text{if} \quad a_n > 0 \quad \text{and} \quad n \geqq 2\,,$$

and by letting

$$a_0 + a_1 x \geqq 0 \quad \text{if} \quad a_0 \geqq 0 \quad \text{and} \quad a_1 \geqq 0\,,$$

we get an example of a lattice-ordered integral domain that cannot be embedded as an order-convex sub-l-ring of a lattice-ordered field.

Generalizing a question posed by Conrad and Dauns [3] for l-fields, we may also ask:

(4) *Under what conditions on an l-ring can its order be extended to a linear order?*

I am able to show that *the order on any lattice-ordered integral domain A in which $a^2 \geqq 0$ for every a in A can be extended to a linear order*, but the general question remains open.

References

[1] *Benali Benzaghou:* Algébres de Hademand. Bull. Soc. Math. France *98* (1970), 209—252.
[2] *G. Birkhoff and R. S. Pierce:* Lattice-ordered rings. An. Acad. Brasil. Ci. *28* (1956), 41—69.
[3] *P. Conrad and J. Dauns:* An embedding theorem for lattice-ordered fields. Pacific J. Math. *30* (1969), 385—398.
[4] *J. E. Diem:* A radical for lattice-ordered rings. Pacific J. Math. *25* (1968), 71—82.
[5] *L. Fuchs:* Partially ordered algebraic systems. Pergammon Press, London, 1963.
[6] *D. G. Johnson:* A structure theory for a class of lattice-ordered rings. Acta Math. *104* (1960), 163—215.
[7] *S. A. Steinberg:* Lattice-ordered rings and modules. Thesis, University of Illinois, Urbana, Illinois, 1970.

HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA