V. Kannan; M. Rajagopalan
On rigidity and groups of homeomorphisms

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 231--234.

Persistent URL: http://dml.cz/dmlcz/700735

ON RIGIDITY AND GROUPS OF HOMEOMORPHISMS

V. KANNAN and M. RAJAGOPALAN

Madurai

If X is an object of a category, we denote by $A(X)$ the group of all automorphisms of X. Thus for example, if X is a topological space, $A(X)$ is the group of all homeomorphisms on X. In this paper, we investigate $A(X)$ for objects in the categories of topological spaces and apply the results to obtain corresponding theorems in the categories of lattices, Boolean algebras, semigroups, graphs, and some other related categories. The groups of homeomorphisms have been investigated by J. de Groot [7] (see also [1], [5], [8] and [9]), the groups of automorphisms of lattices by G. Birkhoff [4] (see also [15]) and the groups of automorphisms of Boolean algebras by M. Katětov [13] and J. de Groot [7] (see also [12], [17]). Here we solve some open problems on this topic, generalize many known results and give different proofs for several results. The following are some of the main theorems of this paper:

Theorem 1. *Every zero-dimensional Hausdorff space is a subspace of a compact rigid zero-dimensional Hausdorff space.*

The above theorem gives a new proof for the solution of Birkhoff's problem 74 [3] and considerably generalizes the results of [12], [14] and [17].

The following theorem supplements the main theorems of J. de Groot [7]. Unlike his proof, which uses the theory of graphs, here we use only the results in topology. J. de Groot's results on graphs are then deduced as corollaries.

Theorem 2. *Let X be any Hausdorff space and G any group. Then there exists a topological space X^* having X as a closed subspace such that $A(X^*)$ is isomorphic to G. Further, X^* can be chosen so that:*

(i) *X^* is connected and locally connected.*

(ii) *Under some mild conditions on X, X and X^* have the same cardinality.*

We have that X^* is sequential if and only if X is. If X is Tychonoff then we can choose X^* to be compact and connected. If X is normal, then X^* can be chosen to be compact, connected, and of the same positive dimension as X.

If $\mathscr{A}$ is the category of all partially ordered topological spaces and if X is an object of $\mathscr{A}$, then the group $A(X)$ is a partially ordered group in a natural way.

Theorem 3. *Given a partially ordered topological space X and a partially ordered group G, we can find a partially ordered space X^* which is an extension of X such that $A(X)$ and G are isomorphic as partially ordered groups.*

Definition. A Hausdorff space X is said to be *chaotic* if any two distinct point x, $y \in X$ have neighbourhoods U_x and U_y, respectively, such that no two non-empty open subspaces of U_x and U_y are homeomorphic.

Theorem 4. (a) *Every Hausdorff k-space is a closed subspace of a chaotic space of the same cardinality.*

(b) *There exists a proper class of completely normal connected locally connected chaotic spaces.*

After writing this paper, we found that E. S. Berney [2] has announced that the real line contains a chaotic subspace, answering in the affirmative parts (a) and (b) of the question in [16]. However, our theorem answers the question completely and the two proofs are entirely different. We are thankful to E. S. Berney for having sent us a preprint of his results.

J. de Groot [7] was the first to construct an example of a Hausdorff space in which every continuous self-map is either identity or constant. H. Herrlich [10] used this result to construct pathological reflections. In [11] he poses the following problem:

Does there exist a proper class of Hausdorff spaces such that any continuous map between any two of them is either constant or identity?

We answer this question in the affirmative. Our example combines another pathology also with this. For, we have

Theorem 5. *Let $\mathscr{T}$ be the category of all topological spaces with continuous maps as morphisms. Then there exists a full subcategory $\mathscr{A}$ of $\mathscr{T}$ such that*

(i) *$\mathscr{A}$ is not small.*

(ii) *Every object in $\mathscr{A}$ is a connected Hausdorff space with a dispersion set of two points.*

(iii) *Every morphism in $\mathscr{A}$ is either constant or identity.*

Generalizing a result of G. Birkhoff [4] we show that

Theorem 6. *Every lattice is a sublattice of a lattice having a given group of automorphisms. If the lattice is distributive, then the extension can also be chosen to be distributive.*

We give a proof via topology to the following theorem of J. de Groot [7] and G. Sabidussi [18].

Theorem 7. *Any group can be realized as the automorphism group of a graph.*

J. de Groot has raised the following question in [7]: What can we say about the cardinality of rigid Boolean algebras? Here we give a complete answer, assuming the generalized continuum hypothesis.

Theorem 8. *Let m be a cardinal. Then there exists a rigid Boolean algebra of cardinality m if and only if m is uncountable, or $m \leqq 2$.*

This theorem improves a result of [14], which answers a weaker question posed in [8].

All rigid Boolean algebras constructed hitherto are, to the best of our knowledge, not σ-complete. M. Katětov [13] asked whether there exist σ-complete rigid Boolean algebras. We answer this in the affirmative.

Theorem 9. *Any Boolean algebra can be embedded in a rigid σ-complete Boolean algebra.*

The proofs will appear elsewhere.

References

[1] *R. D. Anderson:* Zero-dimensional compact groups of homeomorphisms. Pacific J. Math. *7* (1957), 797—810.

[2] *E. S. Berney:* A separable metric space which is chaotic. Notices Amer. Math. Soc. *18* (1971), 261—262.

[3] *G. Birkhoff:* Lattice theory. New York, 1945.

[4] *G. Birkhoff:* On groups of automorphisms. Rev. Un. Mat. Argentina *11* (1945), 155—157.

[5] *N. J. Fine and G. E. Schweigert:* On the groups of homeomorphisms of an arc. Ann. of Math. *62* (1955), 237—253.

[6] *J. de Groot and R. J. Wille:* Rigid continua and topological group-pictures. Arch. Math. *9* (1958), 441—446.

[7] *J. de Groot:* Groups represented by homeomorphism groups I. Math. Ann. *138* (1959), 80—102.

[8] *J. de Groot and R. H. Mc Dowell:* Auto-homeomorphism groups of 0-dimensional spaces. Compositio Math. *15* (1963), 203—209.

[9] *J. de Groot and M. A. Maurice:* On the existence of rigid compact ordered spaces. Proc. Amer. Math. Soc. *19* (1958), 844—846.

[10] *H. Herrlich:* On the concept of reflections in general topology. Contributions to Extension Theory of Topological Structures (Proc. Sympos., Berlin, 1967). Deutsch. Verlag Wissensch., Berlin, 1969, 105—114.

[11] *H. Herrlich:* Categorical topology. General Topology and its Applications *1* (1971), 1—15.

[12] *B. Jonsson:* A Boolean algebra without proper automorphisms. Proc. Amer. Math. Soc. *2* (1951), 766—770.

[13] *M. Katětov:* Remarks on Boolean algebras. Colloq. Math. *2* (1951), 229—235.

[14] *F. W. Lozier:* A class of compact rigid 0-dimensional spaces. Canad. J. Math. *21* (1969), 817—821.

[15] *K. D. Magill, Jr.:* The lattice of compactifications of a locally compact space. Proc. London Math. Soc. *18* (1968), 231—244.
[16] *E. D. Nix:* Chaotic spaces. Bull. Amer. Math. Soc. *76* (1970), 975—976.
[17] *L. Rieger:* Some remarks on automorphisms of Boolean algebras. Fund. Math. *38* (1951), 209—216.
[18] *G. Sabidussi:* Graphs with given infinite group. Monatsh. Math. *64* (1960), 64—67.

MADURAI UNIVERSITY, MADURAI