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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 259--261.

Persistent URL: http://dml.cz/dmlcz/700729

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All partial orderings $<$ (*"strictly precedes"*) are to satisfy $p < q < p \Rightarrow p = q$; the converse implication may, but need not, hold – indeed a point which strictly precedes itself will be called *singular*. The reflexive relation $\leqq$ (*"precedes"*) is associated with $<$ in the usual way ($p \leqq q$ iff $p < q$ or $p = q$); and we write

$$L(q) = \{x \mid x < q\}, \quad L[Q] = \bigcup_{q \in Q} L(q).$$

A non-void subset D of a partially ordered set is *directed* (resp. *strictly directed*) if, given two points in D, there exists a point in D succeeding (resp. strictly succeeding) both; D is a *strict ideal* if it is strictly directed and contains all the predecessors of each of its points. Call $L[Q]$ the set *generated* by Q: then every strictly directed set generates a strict ideal, and every strict ideal generates itself. Any set of the form $L(q)$ is called a *principal strict ideal.*

Borrowing a term from Michael [2], we call a partially ordered set a *cushion* if it satisfies any of the following equivalent conditions:

(a) *Every point has a strict predecessor; and, whenever* p_1, $p_2 < r$, *there exists* q *satisfying* $p_1, p_2 < q < r$.

(b) *Every directed subset generates a strict ideal.*

(c) *Every principal strict ideal is a strict ideal.*

We equip each cushion with the topology determined by the base $\{\langle p, q]\}_{p<q}$, where $\langle p, q] = \{x \mid p < x \leqq q\}$. The singular points of a cushion are then its isolated points, and it is Hausdorff if and only if $p = q$ whenever $L(p) = L(q)$. A subset S of the cushion X is called a *subcushion* of X if S (with the restricted ordering) is itself a cushion whose topology coincides with its topology as a subspace of X. A function f between cushions is called a *cushion map* if it is continuous and $p < q$ implies $f(p) < f(q)$.

Example 1. Let E be a normal T_1-space. (Somewhat weaker separation axioms are in fact sufficient.) Let cE denote the collection of its open subsets other than $\emptyset$ and E, ordered by putting $U < V$ iff $U^- \subset V$. Then cE is a Hausdorff cushion which is non-singular if and only if E is connected. If E, F are normal T_1-spaces and $\theta : E \to F$ is a closed continuous surjection, then $c\theta : cF \to cE$, where $c\theta(W) = \theta^{-1}[W]$, is a cushion map.

Example 2. Let $\mathbf{R}^-$ denote the non-positive real numbers and M be a pseudometric space. Let kM denote the set $M \times \mathbf{R}^-$, ordered by putting $(m_1, r_1) < (m_2, r_2)$ iff $d(m_1, m_2) < r_2 - r_1$. Then kM is a non-singular cushion which is Hausdorff if and only if M is Hausdorff (i.e., metric). If M, N are pseudometric spaces and the function $\varphi : M \to N$ satisfies $d(\varphi(m_1), \varphi(m_2)) < \lambda\, d(m_1, m_2)$ for some fixed positive number λ, then the function $k_\lambda \varphi : kM \to kN$, where $k_\lambda \varphi(m, r) = (\varphi(m), \lambda r)$, is a cushion map.

A cushion in which every strict ideal is principal is called *complete*. Complete cushions have some pleasant properties. Let us, for instance, say that a net $(s_d)_{d \in D}$ (on the directed set D) in a partially ordered space is *increasing* if $d \leqq e$ implies $s_d \leqq s_e$. Then a cushion X is complete (resp. Hausdorff) if and only if every increasing net in X has at least (resp. at most) one limit point. Again: every closed subcushion of a complete cushion is complete, and every complete subcushion of a Hausdorff cushion is closed. The results we shall establish here are two more specific ones.

Theorem 1. *The cushion cE is complete if and only if the topological space E is compact.*

Proof. Assume first that E is compact, and let $\mathscr{P}$ be an ideal in cE; then $P^* = \bigcup_{P \in \mathscr{P}} P$ is open and non-void. Suppose Q is a non-void open set such that $Q^- \subset P^*$. Since $\mathscr{P}$ covers Q^-, so does some finite subcollection $\{P_1, \ldots, P_m\}$ of $\mathscr{P}$. Since $\mathscr{P}$ is directed, some member of $\mathscr{P}$ contains all the P_i and hence contains Q^-. It follows that $Q \in \mathscr{P}$. Since $E \notin \mathscr{P}$, this argument (with $Q = E$) shows that $P^* \neq E$; so $P^* \in cE$. It also shows that $L(P^*) \subset \mathscr{P}$. On the other hand, if $P \in \mathscr{P}$, then $P < P' \in \mathscr{P}$ for some P', so that $P^- \subset P' \subset P^*$: therefore $P \in L(P^*)$. It follows that $\mathscr{P} = L(P^*)$, and hence that cE is complete.

To prove the converse, assume E has an open cover $\mathscr{U}$ with no finite refinement. Let $\mathscr{V}$ denote the collection of all non-void sets expressible as finite unions of members of $L[\mathscr{U}]$. Then $\mathscr{V}$ is a directed subset of cE which fails to generate a principal strict ideal; for if $L[\mathscr{V}] = L(W)$, where $W \in cE$, then (since $\mathscr{V}$ is actually strictly directed) each member of $\mathscr{V}$ is a subset of W, contradicting the fact that $\mathscr{V}$ covers E.

Theorem 2. *The cushion kM is complete if and only if the pseudometric space M is complete.*

Proof. Suppose M is a complete pseudometric space and P is a strict ideal in kM. Let

$$r^* = \sup\{r \mid (x, r) \in P \text{ for some } x \in M\},$$

and choose $x_0, x_1, \ldots$ in M such that $(x_n, r^* - 2^{-n}) \in P$ for each n. Then (x_n) is a Cauchy sequence; and $P = L(x^*, r^*)$, where x^* is a limit of (x_n).

Conversely, suppose that kM is a complete cushion and (y_n) is a Cauchy sequence. Define

$$s_n = -2 \sup_{k \geqq 0} d(y_n, y_{n+k}) .$$

The set $\{(y_0, s_0), (y_1, s_1), \ldots\}$ is directed and therefore generates a strict ideal: call this $L(q)$. Then q must be of the form $(y, 0)$, and y must be a limit of (y_n).

If X is a complete Hausdorff cushion and $f: X \to X$ a cushion map satisfying $a < f(a)$ for some a, then f has a fixed point. (To construct it, put $a_0 = a$, $a_{n+1} = f(a_n)$. If the directed set $\{a_0, a_1, \ldots\}$ generates $L(b)$, then $f(b) = b$.) Theorem 2 shows that this result includes the Banach fixed-point theorem: for if $\varphi: M \to M$ satisfies $d(\varphi(m_1), \varphi(m_2)) < \lambda\, d(m_1, m_2)$, where $\lambda < 1$, and if m is any point of M, then (m, r) strictly precedes $k_\lambda\, \varphi(m, r)$ in the cushion kM for all sufficiently large $-r$; and if $k_\lambda \varphi$ has a fixed point, so has φ. (It may be noted that, working with reflexive orderings, one obtains, instead of propositions about the (complete) cushion kM, closely analogous propositions about (Dedekind σ-complete) ordered sets. See [1].)

References

[1] *R. DeMarr:* Partially ordered spaces and metric spaces. Amer. Math. Monthly *72* (1965), 628–631.

[2] *E. Michael:* Yet another note on paracompact spaces. Proc. Amer. Math. Soc. *10* (1959), 309–314.

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