Luděk Kučera; Aleš Pultr

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THE CATEGORY OF COMPACT HAUSDORFF SPACES IS NOT ALGEBRAIC IF THERE ARE TOO MANY MEASURABLE CARDINALS[1])

L. KUČERA and A. PULTR

Praha

At the 1st Prague Topological Symposium in 1961, J. R. Isbell, who proved before ([5]) that if there is no measurable cardinal the category **Comp** of compact Hausdorff spaces and their continuous mappings is algebraic (i.e., fully embeddable into a category of algebras), put a question whether this holds for the category of all topological spaces. It was positively answered by Hedrlín and Pultr in [2] (communicated at the 2nd Prague Topological Symposium, see also [3]) under a weaker assumption

(M) There is a cardinal $\mathfrak{a}$ such that every ultrafilter closed under intersection of $\mathfrak{a}$ elements is trivial.

In the quoted paper and in further ones (see, e.g., [6]) it was, moreover, proved that every concretizable category which was "constructive" in some sense was algebraic. Since, on the other hand, there was no example of a non-algebraic concretizable category known, the conjecture arose that every concretizable category was algebraic. And this is really the case, by a result of Hedrlín and Kučera proved in 1969 (still unpublished) – under the assumption (M).

In this note, we want to communicate that under non (M), however, the situation is entirely different. Namely, e.g. the categories $\mathbf{Set}^{op}$ and **Comp** become non-algebraic. Full proofs of these facts will be given in a longer forthcoming paper. Here we will sketch them roughly.

Theorem 1. *Under* non (M), $\mathbf{Set}^{op}$ *is not algebraic.*

The proof of this is based on the following

Lemma. *If there exists an $\mathfrak{a}$-additive non-trivial two-valued measure on a set X and if $F : \mathbf{Set} \to \mathbf{Set}$ is a contravariant faithful functor such that* card $F(1) \leqq \mathfrak{a}$, *then there is a mapping $\mu : F(X) \to F(1)$ such that* (1) *for every $\xi : 1 \to X$, $\mu \neq F(\xi)$, and* (2) *if* card $A \leqq \mathfrak{a}$, *then for every $\alpha : A \to X$ there is a $\xi : 1 \to X$ with $\mu \circ \alpha = F(\xi) \circ \alpha$.*

We cannot go here into the proof of this statement. To elucidate what it says, let us only point out that the measure itself has the required properties if F is the

[1]) Preliminary communication.

contravariant power set functor. Let us show now how the theorem follows: Suppose **Set**^{op} is algebraic. Then, by [1], there is a full embedding $\Phi : \mathbf{Set}^{op} \to \mathfrak{R}$ where $\mathfrak{R}$ is the category of sets with binary relations and relation preserving mappings. Denote by U the natural forgetful functor $\mathfrak{R} \to \mathbf{Set}$. Put $F = U \circ \Phi$. If non (M) holds, there is a mapping $\mu : F(X) \to F(1)$ with the properties described above. By (2), μ carries a morphism from $\Phi(X)$ into $\Phi(1)$ and, by (1), this morphism is not in the image of Φ, in a contradiction with the assumption that Φ is full.

Theorem 2. *Under* non (M), **Comp** *is not algebraic.*

The proof goes, roughly, as follows: If a concrete category $(\mathfrak{R}, U)$ is fully embeddable into a category of algebras in such a way that the underlying set of the algebra corresponding to an object always contains the original underlying set and that the homomorphisms corresponding to morphisms are extensions of their original underlying mappings, it is not difficult to prove that then a category obtained from $(\mathfrak{K}, U)$ endowing the objects by relational systems on their underlying sets, and taking only those morphisms the underlying mappings of which preserve them, is again algebraic.

Using Pontrjagin duality we can prove that **Set**^{op} is fully embeddable into the category thus obtained from the category of compact abelian groups adding two unary relations. Consequently, it is fully embeddable into the category obtained from **Comp** by adding one ternary and two binary relations. Thus, **Comp** endowed by the natural forgetful functor cannot be, by Theorem 1, embedded into a category of algebras in the way described above. But, by Theorem 1.8 in [7], it cannot be then fully embedded into a category of algebras at all.

Let us remark that Theorem 2 in a way completes the answer to the mentioned Isbell's problem on the category of topological spaces. Namely, we have

Corollary. *The following three statements are equivalent*:

(a) (M).

(b) **Comp** *is algebraic.*

(c) *The category of topological spaces and their continuous mappings is algebraic.*

As an immediate consequence of Theorems 1 and 2 we obtain the following

Corollary. *Under* non (M) *the following categories are non-algebraic*:

the category of complete Boolean algebras and their complete homomorphisms,
the category of topological spaces and their closed continuous mappings,
the category of uniform spaces and their uniformly continuous mappings.

Several further categories (topological spaces with open or open continuous mappings, sets with mappings onto etc.) can be shown to be non-algebraic under non (M) using more complicated considerations. In contrast with the situation with (M), under non (M) the property to be algebraic becomes a very special one. In fact, we do not know at the moment an example of a nice category without a small left adequate (see [4], [5]) which would be algebraic under non (M).

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CHARLES UNIVERSITY, PRAHA