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TOPOLOGICAL REPRESENTATIONS OF MEASURABLE SPACES

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1. Measurable spaces

1.1. A measurable space is a pair $(X, \mathscr{B})$, where X is a set and $\mathscr{B}$ is a σ-algebra of subsets of X. By a topological representation of a measurable space $(X, \mathscr{B})$ we mean a completely regular Hausdorff topology τ on X such that $\mathscr{B}$ coincides with the family of Baire sets generated by τ (i.e. the smallest family of subsets of X containing the zero sets of τ-continuous real valued functions and closed under countable unions and countable intersections).

All proofs are omitted, and the references (which would accompany the proofs) are restricted. However, we point out that the work of Z. Frolík, much of which is surveyed in [1], is central to several of our demonstrations.

1.2. Notation. Let $(X, \mathscr{B})$ be a measurable space. We denote by $B(X, \mathscr{B})$ the space of measurable functions from $(X, \mathscr{B})$ into $(\mathbf{R}, \mathscr{B}_0)$, where $\mathbf{R}$ denotes the real line and $\mathscr{B}_0$ its family of Baire sets (equivalently, Borel sets). $B^*(X, \mathscr{B})$ denotes the subspace of bounded functions in $B(X, \mathscr{B})$.

If $(X, \mathscr{B})$ is topologically representable, then both $B(X, \mathscr{B})$ and $B^*(X, \mathscr{B})$ form lattice-ordered rings under the pointwise algebraic and lattice operations.

1.3. Theorem. *If $(X, \mathscr{B})$ and $(Y, \mathscr{F})$ both admit realcompact representations, then the following are equivalent:*

1) *$(X, \mathscr{B})$ is measurably isomorphic to $(Y, \mathscr{F})$,*
2) *$B(X, \mathscr{B})$ is ring isomorphic to $B(Y, \mathscr{F})$,*
3) *$B(X, \mathscr{B})$ is lattice isomorphic to $B(Y, \mathscr{F})$,*
4) *$B(X, \mathscr{B})$ is multiplicative semigroup isomorphic to $B(Y, \mathscr{F})$.*

This theorem along with the fact that every Baire function on a completely regular space X has a unique extension to the Hewitt realcompactification of X (due to P. R. Meyer – 1961, unpublished – in the more precise form that a Baire function of class α extends uniquely to one of class α on the realcompactification) allows us to apply the argument given in [4] to obtain the following:

1.4. Theorem. *If $(X, \mathscr{B})$ and $(Y, \mathscr{F})$ are topologically representable spaces, then the following are equivalent:*

1) *$B(X, \mathscr{B})$ is ring isomorphic to $B(Y, \mathscr{F})$,*
2) *$B(X, \mathscr{B})$ is lattice isomorphic to $B(Y, \mathscr{F})$,*
3) *$B(X, \mathscr{B})$ is multiplicative semigroup isomorphic to $B(Y, \mathscr{F})$.*

1.5. Theorem. *If $(X, \mathscr{B})$ and $(Y, \mathscr{F})$ are topologically representable and have the property that for all $x \in X$ and $y \in Y$ we have $\{x\} \in \mathscr{B}$ and $\{y\} \in \mathscr{F}$, then the following are equivalent:*

1) *$(X, \mathscr{B})$ is measurably isomorphic to $(Y, \mathscr{F})$,*

2), 3, 4) *$B(X, \mathscr{B})$ is ring (lattice, multiplicative semigroup) isomorphic to $B(Y, \mathscr{F})$,*

5), 6), 7), 8) *$B^*(X, \mathscr{B})$ is ring (lattice, multiplicative semigroup, Banach space) isomorphic to $B^*(Y, \mathscr{F})$.*

1.6. The results of Theorems 1.3, 1.4, and 1.5 can be demonstrated for the spaces of Baire functions of class α for each countable ordinal α and explicit relations between the various automorphisms of these spaces and the measurable automorphisms of the underlying measurable space can be given (see [6]).

1.7. Let $(X, \mathscr{B})$ be a topologically representable space and let $\hat{X}$ denote the compactification of X (considering X with the weak topology generated by $B^*(X, \mathscr{B})$) determined by $B^*(X, \mathscr{B})$. The points of $\hat{X}$ are in one to one correspondence with the $\mathscr{B}$-ultrafilters on X, each point $p \in \hat{X}$ corresponding to the unique $\mathscr{B}$-ultrafilter U_p on X which converges to it [3]. The $\mathscr{B}$-ultrafilters are in one to one correspondence with the maximal ring ideals in $B(X, \mathscr{B})$ [6]. Thus for each point $p \in \hat{X}$ there corresponds a maximal ring ideal M_p in $B(X, \mathscr{B})$ defined by

$$M_p = \{f \in B(X, \mathscr{B}) : Z(f) \in U_p\},$$

where $Z(f) = \{x : f(x) = 0\}$.

We have the following analog of the Gelfand-Kolmogoroff theorem:

1.8. Theorem. *If $(X, \mathscr{B})$ is topologically representable, then each maximal ring ideal M_p $(p \in \hat{X})$ in $B(X, \mathscr{B})$ is of the form*

$$M_p = \{f \in B(X, \mathscr{B}) : p \in \mathrm{cl}_{\hat{X}} Z(f)\},$$

where $\mathrm{cl}_{\hat{X}} Z(f)$ denotes the closure in $\hat{X}$ of $Z(f)$.

In [6] this theorem is demonstrated for the rings of Baire functions of class α for every countable ordinal α, thus giving a generalization of the Gelfand-Kolmogoroff theorem.

1.9. Theorem. *If $(X, \mathscr{B})$ admits a realcompact representation, then the intersection of all free maximal ring ideals in $B(X, \mathscr{B})$ is the ideal of functions with finite support.*

1.10. Definition. A completely regular Hausdorff space X is called a *ZS-space* if it is obtainable from the zero sets of its Stone-Čech compactification βX by Suslin's operation (A), i.e. of the form

$$\bigcup_{\sigma \in \mathbf{N}^{\mathbf{N}}} \bigcap_{s < \sigma} Z(f_s),$$

where $\mathbf{N} = \{1, 2, \ldots\}$, $s < \sigma$ means that s is a restriction of σ, and the f_s are continuous real valued functions on βX.

The class of ZS-spaces will be denoted $\mathscr{S}$ and its subclass of spaces which are Baire subsets of their Stone-Čech compactifications by $\mathscr{Z}$ (in view of the fact that they are zero sets of Baire functions on βX).

Each space in $\mathscr{S}$ is Lindelöf and therefore realcompact. Thus the preceding theorems apply to representations in this class of spaces.

1.11. Theorem. *If $(X, \mathscr{B})$ admits a representation in $\mathscr{S}$, then the following are equivalent:*

1) *The Baire class hierarchy is not strictly increasing for all countable ordinals for some representation of $(X, \mathscr{B})$ in $\mathscr{S}$,*

2) *The Baire class hierarchy is not strictly increasing for all countable ordinals for every representation of $(X, \mathscr{B})$ in $\mathscr{S}$,*

3) *Every $B \in \mathscr{B}$ is multiplicative Baire class 1 for some representation of $(X, \mathscr{B})$ in $\mathscr{S}$,*

4) *Every $B \in \mathscr{B}$ is multiplicative Baire class 1 for every representation of $(X, \mathscr{B})$ in $\mathscr{S}$,*

5) *$\mathscr{B}$ is invariant under the operation (A),*

6) *Some representation of $(X, \mathscr{B})$ in $\mathscr{S}$ does not contain a compact perfect subset,*

7) *Every representation of $(X, \mathscr{B})$ in $\mathscr{S}$ does not contain a compact perfect subset,*

8) *$B(X, \mathscr{B})$ is the space of all real valued functions which are continuous in the weak topology generated on X by $B(X, \mathscr{B})$,*

9) *The weak topology generated on X by $B(X, \mathscr{B})$ has the Lindelöf property.*

1.12. Theorem. *If $(X, \mathscr{B})$ admits a metrizable representation in $\mathscr{S}$, then every representation of $(X, \mathscr{B})$ in $\mathscr{S}$ is metrizable. All of these representations are separable absolute analytic topologies (analytic subsets of their metric completions).*

2. Spaces of representations

2.1. We now turn to the analysis of spaces of representations of a fixed measurable space.

Let $(X, \mathscr{B})$ be a measurable space admitting a separable metrizable absolute Borel set representation. Let T denote the space of compact representations of $(X, \mathscr{B})$ (each of which must be metrizable by Theorem 1.12). E. R. Lorch has introduced a topology on T for which the automorphism group G of $(X, \mathscr{B})$ acts naturally as a homeomorphism group of T [8]. He and H. Tong have characterized the discrete points of this topology as the finite dimensional topologies in T and related this to the corresponding uniformities (mentioned above) on G [10].

A variant approach is to fix a class of topological spaces, introduce the equivalence relation of Baire isomorphism, and attempt to describe the resulting equivalence classes. In Lorch's setting above the orbits of the action of G on T form the single equivalence class in the class of all compact (uncountable) metrizable spaces. For the class of separable metrizable absolute Borel sets it is known that cardinality alone determines the Baire isomorphism classes [7]. For the class of all metrizable absolute Borel sets A. H. Stone has given a complete set of topological invariants of Baire isomorphism (see [13] and [14]).

For the class of analytic non-Borel subsets of separable completely metrizable spaces the problem is open. It is known that if there exists a coanalytic subset of the real line of power of the continuum which does not contain a homeomorph of the Cantor set, then there exist two analytic non-Borel subsets of separable completely metrizable spaces which are not Baire isomorphic [11]. The hypothesis of this result is consistent [2] and independent [12] of the axioms of Zermelo-Fraenkel set theory including the axiom of choice. It is implied by Gödel's axiom of constructibility [2] and its negation is implied by the existence of a measurable cardinal [12]. It is also known that any two universal analytic non-Borel subsets of the plane are Baire isomorphic [11].

Since analytic subsets of separable completely metrizable spaces belong to $\mathscr{S}$, this is part of the more extensive problem of describing all of the Baire isomorphism classes in $\mathscr{S}$. Toward this goal we give $\mathscr{S}$ the structure of a category by defining the morphisms between two spaces to be perfect maps (i.e. closed, continuous, and the inverses of points are compact). Note that $\mathscr{X}$ is then a full subcategory.[1]

2.2. Theorem. 1) *A space in $\mathscr{S}$ is a projective object if and only if it is extremally disconnected.*

2) *Every space in $\mathscr{S}$ has a projective resolution. If a space is in $\mathscr{X}$, then its projective resolutions are also in $\mathscr{X}$.*

3) *The Baire isomorphism class of every projective space in $\mathscr{X}$ contains a compact space.*

4) *Every projective space X in $\mathscr{X}$ has the property that*

$$\bigcup_{\tau\in K(X,\mathscr{B})} C(X_\tau)$$

is uniformly dense in $B^*(X, \mathscr{B})$, where $\mathscr{B}$ is the Baire σ-algebra of X, $K(X, \mathscr{B})$ is the set of compact representations of $(X, \mathscr{B})$, and $C(X_\tau)$ is the space of τ-continuous real valued functions on X.

Part 4) of this theorem for a Borel subset of a separable completely metrizable space is due to E. R. Lorch and H. Tong [9].

A Baire quotient map from a space X into a space Y is one with the property that the inverse of a set in Y is a Baire set in X if and only if the set is a Baire set in Y. It follows from parts 2) and 3) of the above theorem that every space in $\mathscr{X}$ is the Baire quotient image of a compact space.

References

[1] *Z. Frolík:* A survey of the separable descriptive theory of sets and spaces. Czechoslovak Math. J. *20* (1970), 406—467.

[2] *K. Gödel:* The consistency of the axiom of choice and the generalized continuum-hypothesis. Proc. Nat. Acad. Sci. U.S.A. *24* (1938), 556—557.

[3] *H. Gordon:* Compactifications defined by means of generalized ultrafilters. Ann. Mat. Pura Appl. *86* (1970), 15—23.

[4] *M. Henriksen:* On the equivalence of the ring, lattice, and semigroup of continuous functions. Proc. Amer. Math. Soc. *7* (1956), 959—960.

[5] *J. E. Jayne:* Descriptive set theory in compact spaces. Notices Amer. Math. Soc. *17* (1970), 268.

[6] *J. E. Jayne:* Spaces of Baire functions, Baire classes, and Suslin sets. Doctoral dissertation, Columbia University, New York, 1971.

[7] *K. Kuratowski:* Topology, Vol. I. Academic Press, New York, 1966.

[8] *E. R. Lorch:* On compact metric spaces and the group of Baire equivalences. Studia Math. *31* (1968), 243—252.

[9] *E. R. Lorch and H. Tong:* Continuity of Baire functions and order of Baire sets. J. Math. Mech. *16* (1967), 991—996.

[10] *E. R. Lorch and H. Tong:* Caractérisation de certain topologies compactes. Enseignement Math. *15* (1969), 153—157.

[11] *A. Maitra and C. Ryll-Nardzewski:* On the existence of two analytic non-Borel sets which are not isomorphic. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *18* (1970), 177—178.

[12] *R. M. Solovay:* On the cardinality of Σ_2^1 sets of reals. Foundations of Mathematics, Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel. Springer-Verlag, Berlin, 1969, 58—73.

[13] *A. H. Stone:* On σ-discreteness and Borel isomorphism. Amer. J. Math. *85* (1963), 655—666.

[14] *A. H. Stone:* Non-separable Borel sets II. (To appear.)

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