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ON SOME TOPOLOGICAL GAMES

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The present contribution treats on some game-theoretic methods in the general topology.

The term „topological games" was introduced by C.Berge [2] (cf. also [3]) to distinguish those positional games whose rules are continuous (as multivalued functions). Here and also in [25], [26] and [27] it is proposed to use the term „topological games" without that limitation of rules. The topological character of games considered here is determined by topological objects and topological operations being involved in actions of players and in calculations of plays' results.

The first paper describing a topological game, although no game theory notion is used, is due to W.Sierpiński [24] (cf. Section 1 below). Topological games with the transfinite length of plays were used for the first time by A.V.Arhangel'skiĭ [1], however without using notions of game theory as well. The most widespread known topological games are the Banach-Mazur game over the unit interval ([22], [23]) and the binary game over the Cantor discontinuum ([4], [11], [15], [18], [19], [20], [21], [29]).

Each one of eight games considered below is an infinite positional two-person win-lose game with perfect information (the theory of those games arose independently in Poland ([20], [21]) and in the U.S.A. ([11])). There are two players, player I and player II, which alternately choose certain objects, say A_1, B_1, A_2, B_2, ..., connected with a given topological space (e.g., points, subsets, covers). Player I chooses A_1 and each choice is made with complete information about previous choices. The choice of A_n (B_n) is the n^{th} move of player I (resp. player II) and $\langle A_1, B_1, A_2, B_2, \ldots \rangle$ is a play of the game. The result of a play is either the win or the loss for a player. A strategy of a player is a function which prescribes the next move of him provided that a (finite) sequence of his opponent's moves is given. A strategy of a player is a winning one if, applying that one, he wins each play. It turns out that a winning strategy of a player describes a topological property of an approximation type.

1. Smooth sets

A family $\mathcal{E}$ of subsets of a set Y is said to be a L-family if
(a) $E \in \mathcal{E}$ and $E \subset X \subset Y$ implies $X \in \mathcal{E}$, and
(b) $\bigcup\{X_n : n \in N\} \in \mathcal{E}$ implies $X_n \in \mathcal{E}$ for some $n \in N$.

Examples of L-families: 1. $\mathcal{E} = \{E \subset Y: \text{card } E > \aleph_0\}$, 2. $\mathcal{E} = \{E \subset Y: \text{int}_Y \text{cl}_Y E \neq 0\}$, where Y is a Polish space, and 3. $\mathcal{E} = \{E \subset Y: m^*(E) > 0\}$, where m is a Borel measure on Y with $m(Y) = 1$.

Let X be a subset of a space Y and let $\mathcal{E}$ be a L-family of subsets of Y. We define a game $G(X,Y,\mathcal{E})$ as follows. Player I chooses a subset D_1 of X. After that player II chooses a subset E_1 of D_1 such that $E_1 \in \mathcal{E}$ if $D_1 \in \mathcal{E}$ and otherwise he chooses $E_1 = 0$. Assume that $D_1, E_1, \ldots, D_n, E_n$ have been chosen. Then player I chooses a subset D_{n+1} of E_n. After that player II chooses a subset E_{n+1} of D_{n+1} such that $E_{n+1} \in \mathcal{E}$ if $D_{n+1} \in \mathcal{E}$ and otherwise he chooses $E_{n+1} = 0$. Player II wins the play $\langle D_1, E_1, D_2, E_2, \ldots \rangle$ of $G(X,Y,\mathcal{E})$ iff $\bigcap\{\text{cl}_Y D_n : n \in N\} \subset X$.

The game $G(X,Y,\mathcal{E})$, where $Y = R^n$ and $\mathcal{E} = \{E \subset Y: \text{card } E > \aleph_0\}$ was described by W.Sierpiński [24], however, he did not use any notion of game theory. Let us note that the Banach-Mazur game was proposed by S.Mazur in about 1928 (cf. [23], Chapter 6). The pioneer technique of W.Sierpiński was recently extended by C.Dellacherie in [5] and [6], but not involving notions of game theory. The notion of L-family is due to C.Dellacherie ([6], Chapter I, D 13). In [5] and [6] a strategy of player II corresponds to the notion of scraper (in French: le rabotage) and the set X for which player II has a winning strategy in $G(X,Y,\mathcal{E})$ is said to be smooth (Y and $\mathcal{E}$ are fixed). The theorem which follows is just another variant of a theorem of C.Dellacherie ([6], Ch.I, Thm.40), but the proofs are different.

<u>Theorem 1.1.</u> Let X be a Souslin set in Y, let $\mathcal{E}$ be a L-family of subsets of Y, and let us assume that $X \in \mathcal{E}$. Then player II has a winning strategy in $G(X,Y,\mathcal{E})$.

Proof. Let X be a Souslin set in Y such that $X \in \mathcal{E}$, where $\mathcal{E}$ is a L-family of subsets of Y. By Theorem 3.2 below there exists a sequence $\langle \mathcal{E}_1, \mathcal{E}_2, \ldots \rangle$ of countable partitions of X such that $\mathcal{E}_{n+1}$ refines $\mathcal{E}_n$ for each $n \in N$ and such that $\bigcap\{\text{cl}_Y E_n : n \in N\} \subset X$ for each sequence $\langle E_1, E_2, \ldots \rangle$ with $E_{n+1} \subset E_n \in \mathcal{E}_n$, where $n \in N$. We define a strategy s for player II as follows. Let $n \in N$ and let $\langle D_1, \ldots, D_n \rangle$ be a sequence of subsets of X such that $D_1 \supset D_2 \supset \ldots \supset D_n$ and $D_k \in \mathcal{E}$ for each $k \leq n$. Then $D_n = \bigcup\{D_n \cap E : E \in \mathcal{E}_n\}$ and thus there exists a $E_n \in \mathcal{E}_n$ such that

$D_n \cap E_n \in \mathcal{E}$. We set $s(D_1,\dots,D_n) = D_n \cap E_n$. If $n \in N$ and $\langle D_1,\dots,D_n\rangle$ is a sequence of subsets of X such that $D_1 \supset D_2 \supset \dots \supset D_n$ and $D_n \notin \mathcal{E}$, then we set $s(D_1,\dots,D_n) = 0$. If $\langle D_1,E_1,D_2,E_2,\dots\rangle$ is a play of $G(X,Y,\mathcal{E})$ such that $E_n = s(D_1,\dots,D_n)$ for each $n \in N$, then $\bigcap\{cl_Y E_n : n \in N\} \subset X$. Hence s is a winning strategy of player II.

<u>Theorem 1.2.</u> Let X be a subset of an uncountable Polish space Y and let $\mathcal{E} = \{E \subset Y: \text{card } E > \aleph_0\}$. Then
(a) If player I has a winning strategy in $G(X,Y,\mathcal{E})$, then Y-X contains a copy of the Cantor discontinuum.
(b) If player II has a winning strategy in $G(X,Y,\mathcal{E})$, then either X is countable or it contains a copy of the Cantor discontinuum.

The proof of part (a) is similar to that one of (b), and part (b) was proved by W.Sierpiński [24].

A separable metric space which contains no copy of the Cantor discontinuum is called totally imperfect ([14], p.514). Assume that $2^{\aleph_0} = \aleph_\alpha$ for some α. Then every uncountable Polish space Y contains a set X which, together with its complement, is totally imperfect and has the cardinality $2^{\aleph_0}$ (cf. [14], p.514), and thus neither of players has a winning strategy in $G(X,Y,\mathcal{E})$, where $\mathcal{E} = \{E \subset Y: \text{card } E > \aleph_0\}$.

2. Large sets

Let X be a space. We define a game G(X) as follows. Player I chooses an open cover $\mathcal{E}_1$ of X with card $\mathcal{E}_1 \leq 2$. After that player II chooses a $E_1 \in \mathcal{E}_1$. Assume that $\mathcal{E}_1, E_1, \dots, \mathcal{E}_n, E_n$ have been chosen. Then player I chooses an open cover $\mathcal{E}_{n+1}$ of X with card $\mathcal{E}_{n+1} \leq 2$. After that player II chooses a $E_{n+1} \in \mathcal{E}_{n+1}$. Player II wins the play $\langle \mathcal{E}_1, E_1, \mathcal{E}_2, E_2, \dots\rangle$ of G(X) iff $\bigcup\{E_n : n \in N\} = X$.

<u>Theorem 2.1.</u> Let X_1 and X_2 be spaces such that either X_1 is a closed subset of X_2 or X_1 is a continuous image of X_2. Then
(a) If player I has a winning strategy in $G(X_1)$, then he has a winning strategy in $G(X_2)$.
(b) If player II has a winning strategy in $G(X_2)$, then he has a winning strategy in $G(X_1)$.

The proof is easy and thus it is omitted.

According to the preceding theorem we may say that a space X is large (small) if player I (resp. player II) has a winning strategy in G(X).

Theorem 2.2. Player I has a winning strategy in G(X) iff X contains a closed subset F which admits a continuous map onto the Cantor discontinuum.

Hence, in particular, the closed unit interval and the Cantor discontinuum are large spaces.

Proof. ($\Longrightarrow$) Let s be a winning strategy of player I in G(X). We set $s(\emptyset) = \{E(0), E(1)\}$, where $\emptyset$ denotes the void sequence, and $s(E(e_1),\dots,E(e_1,\dots,e_n)) = \{E(e_1,\dots,e_n,0), E(e_1,\dots,e_n,1)\}$ for each $\langle e_1,\dots,e_n\rangle \in \{0,1\}^n$ and $n \in N$. Further, we set $F = \bigcup\{\bigcap\{X-E(e_1,\dots,e_n) : n \in N\} : \langle e_1,e_2,\dots\rangle \in \{0,1\}^N\}$ and $F(e_1,\dots e_n) = F-E(e_1,\dots,e_n)$ for each $\langle e_1,\dots,e_n\rangle \in \{0,1\}^n$ and $n \in N$. Since $(X-E(0)) \cap (X-E(1)) = 0$ and $(X-E(e_1,\dots,e_n,0)) \cap (X-E(e_1,\dots,e_n,1)) = 0$ for each $\langle e_1,\dots,e_n\rangle \in \{0,1\}^n$ and $n \in N$, we have $F = \bigcap\{\bigcup\{F(e_1,\dots,e_n) : \langle e_1,\dots,e_n\rangle \in \{0,1\}^n\} : n \in N\}$. Hence F is closed in X. Let us set $f(x) = \langle e_1,e_2,\dots\rangle$ if $x \in \bigcap\{F(e_1,\dots,e_n) : n \in N\}$. It is easy to check that f is a continuous map from F onto $\{0,1\}^N$.

($\Longleftarrow$) Let f be a continuous map from a closed subset F of X onto $\{0,1\}^N$. We set $B(e_1,\dots,e_n) = \{\langle d_1,d_2,\dots\rangle \in \{0,1\}^N : \langle d_1,\dots,d_n\rangle = \langle e_1,\dots,e_n\rangle\}$, $E(e_1,\dots,e_n) = X-f^{-1}(B(e_1,\dots,e_n))$ and $s(\emptyset) = \{E(0),E(1)\}$ and $s(E(e_1),\dots,E(e_1,\dots,e_n)) = \{E(e_1,\dots,e_n,0), E(e_1,\dots,e_n,1)\}$ for each $\langle e_1,\dots,e_n\rangle \in \{0,1\}^n$ and $n \in N$. It is easy to verify that s is a winning strategy for player I.

By Tietze-Urysohn Extension Theorem (cf. [9], p.67) we get from Theorem 2.2 following

Theorem 2.3. Let X be a normal space. Then player I has a winning strategy in G(X) iff X admits a continuous map onto the closed unit interval [0,1].

One can prove the following zero-dimensional variant of Tietze-Urysohn Extension Theorem: Let f be a continuous map from a closed subset F of a normal space X with dim X = 0 into the Cantor discontinuum C. Then f admits a continuous extension f which maps X into C. Hence and by Theorem 2.2 we have

Theorem 2.4. Let X be a normal space with dim X = 0. Then player I has a winning strategy in G(X) iff X admits a continuous map onto the Cantor discontinuum.

It is easy to prove that each completely regular space X with ind $X > 0$ admits a continuous map onto $[0,1]$. Thus by theorems 2.1 and 2.2 we have

Theorem 2.5. If X is completely regular and player II has a winning strategy in G(X), then ind $X \leq 0$.

Similarly, if X is a normal space and player II has a winning strategy in G(X), then dim $X \leq 0$.

Theorem 2.6. Let X be a subset of the Cantor discontinuum C such that C-X is totally imperfect. Then X admits a continuous map onto C and thus player I has a winning strategy in G(X).

Proof. Let h be a homeomorphism from C onto $C \times C$ and let p be the projection map given by $p(s,t) = s$, where $\langle s,t\rangle \in C \times C$. We set $f(x) = p(h(x))$ for each $x \in X$. Since $(C \times C)-h(X)$ is totally imperfect, we have $(\{s\} \times C) \cap h(X) \neq 0$ for each $s \in C$. Hence $f(X) = C$.

Let us note that in Theorem 2.6 the space C can be replaced by $[0,1]$. To prove that variant of Theorem 2.6 it is sufficient to replace h by a continuous map g from $[0,1]$ onto $[0,1] \times [0,1]$.

Theorem 2.7. If X is a scattered Lindelöf regular space, then player II has a winning strategy in G(X).

The proof proceeds by the transfinite induction with respect to α such that $X^{(\alpha)} = 0$ in the same manner as the one of Theorem 9.3 in [25]. Thus it is omitted.

Theorem 2.8. Let X be a metric separable space. Then player II has a winning strategy in G(X) iff X is (at most) countable.

Proof. $(\Rightarrow)$ Let X be a separable metric space and let s be a winning strategy of player II in G(X). By Theorem 2.5 we have ind X = = 0. Hence X can be considered as a subspace of the Cantor discontinuum C. Now we apply an argument given in [12] (cf. also [4]). Let $\mathcal{B}$ be the family of all clopen subsets B of C with $0 \neq B \neq C$. Then

card $\mathcal{B} = \aleph_0$. We may and do assume that player I, on his n^{th} move, chooses a $B_n \in \mathcal{B}$ and after that player II chooses a $E_n \in \{B_n, C-B_n\}$. We set $X_1 = \bigcap\{s(B): B \in \mathcal{B}\}$ and
$X_{n+1} = \bigcup\{\bigcap\{s(B_1,\dots,B_n,B)-s(B_1,\dots,B_n): B \in \mathcal{B}\}: \langle B_1,\dots,B_n\rangle \in \mathcal{B}^n\}$
for each $n \in N$. We claim that $X \subset \bigcup\{X_n: n \in N\}$. Suppose there exists a $x \in X$ with $x \notin X_n$ for each $n \in N$. Then there is a $B_1 \in \mathcal{B}$ with $x \notin s(B_1)$, there is a $B_2 \in \mathcal{B}$ with $x \notin s(B_1,B_2)$, and so on, i.e., there is a $\langle B_1,B_2,\dots\rangle \in \mathcal{B}^N$ with $x \notin \bigcup\{s(B_1,\dots,B_n): n \in N\}$. However that situation cannot occur because s is a winning strategy of player II. Now we claim that card $X_n \leq \aleph_0$ for each $n \in N$. Since $\mathcal{B}$ separates points of X, it follows that each one of the sets $\bigcap\{s(B): B \in \mathcal{B}\}$ and $\bigcap\{s(B_1,\dots,B_n,B)-s(B_1,\dots,B_n): B \in \mathcal{B}\}$ contains at most one point. Thus card $X \leq \aleph_0$.

($\Leftarrow$) We set $X = \{x_1, x_2, \dots\}$ and define a strategy s of player II as follows: $s(\mathcal{E}_1,\dots,\mathcal{E}_n)$ is an element of $\mathcal{E}_n$ containing x_n. Clearly, s is a winning strategy of player II.

As a corollary to thorems 2.3 and 2.8 we get

<u>Theorem 2.9.</u> Let X be a separable metric space. Then
(a) player I has a winning strategy in G(X) iff X admits a continuous map onto [0,1], and
(b) player II has a winning strategy in G(X) iff card $X \leq \aleph_0$.

As a corollary to theorems 2.1 and 2.9 we have

<u>Theorem 2.10. ([12]).</u> Let X be a discrete space. Then
(a) player I has a winning strategy in G(X) iff card $X \geq 2^{\aleph_0}$, and
(b) player II has a winning strategy in G(X) iff card $X \leq \aleph_0$.

3. Souslin sets

Let X be a subset of a space Y. We define a game G(X,Y) as follows. Player I chooses a countable partition $\mathcal{E}_1$ of X. After that player II chooses a $E_1 \in \mathcal{E}_1$. Assume that $\mathcal{E}_1, E_1, \dots, \mathcal{E}_n, E_n$ have been chosen. Then player I chooses a countable partition $\mathcal{E}_{n+1}$ of E_n. After that player II chooses a $E_{n+1} \in \mathcal{E}_{n+1}$. Player I wins the play $\langle \mathcal{E}_1, E_1, \mathcal{E}_2, E_2, \dots\rangle$ of G(X,Y) iff $\bigcap\{cl_Y E_n: n \in N\} \subset X$.

A subset X of a space Y is said to be a Souslin set in Y (more precisely: a F-Souslin set in Y) if there exists an indexed family

$\{F(k_1,\dots,k_n)\colon \langle k_1,\dots,k_n\rangle \in N^n,\ n\in N\}$ of closed subsets of Y such that $X = \bigcup\{\bigcap\{F(k_1,\dots,k_n)\colon n\in N\}\colon \langle k_1,k_2,\dots\rangle \in N^N\}$.

Theorem 3.1. Player I has a winning strategy in $G(X,Y)$ iff X is a Souslin set in Y.

Theorem 3.1 is an immediate consequence of Theorem 3.2.

Theorem 3.2. The following conditions are equivalent:
(a) X is a Souslin set in Y.
(b) There exists an indexed family $\{E(k_1,\dots,k_n)\colon \langle k_1,\dots,k_n\rangle \in N^n,\ n\in N\}$ of subsets of X such that $\bigcup\{E(k)\colon k\in N\} = X$, $\bigcup\{E(k_1,\dots,k_n,k)\colon k\in N\} = E(k_1,\dots,k_n)$ for each $\langle k_1,\dots,k_n\rangle \in N^n$, $n\in N$, and $\bigcap\{cl_Y E(k_1,\dots,k_n)\colon n\in N\}\subset X$ for each $\langle k_1,k_2,\dots\rangle \in N^N$.
(c) There exists a sequence $\langle \mathcal{E}_1,\mathcal{E}_2,\dots\rangle$ of countable partitions of X such that $\mathcal{E}_{n+1}$ refines $\mathcal{E}_n$ for each $n\in N$, and $\bigcap\{cl_Y E_n\colon n\in N\}\subset X$ for each sequence $\langle E_1,E_2,\dots\rangle$ with $E_{n+1}\subset E_n\in \mathcal{E}_n$ for each $n\in N$.

For the proof of Theorem 3.2 we refer to [27]. Let us note that for Souslin sets in compact Hausdorff spaces one can obtain a game-theoretic characterization related to the technique of complete sequences of covers (cf. [10], Section 9).

4. Webbed spaces

Let X be a locally convex vector space. We define a game $G(X)$ as follows. Player I chooses a sequence $\mathcal{E}_1 = \langle E(1,1),E(1,2),\dots\rangle$ of absolutely convex subsets of X such that $\bigcup\mathcal{E}_1$ absorbs X. After that player II chooses a $k_1\in N$. Assume that $\mathcal{E}_1,k_1,\dots,\mathcal{E}_n,k_n$ have been chosen. Then player I chooses a sequence $\mathcal{E}_{n+1} = \langle E(n+1,1),E(n+1,2),\dots\rangle$ of absolutely convex subsets of X such that $E(n+1,k)\subset E(n,k_n)$ for each $k\in N$ and such that $\bigcup\mathcal{E}_{n+1}$ absorbs $E(n,k_n)$. After that player II chooses a $k_{n+1}\in N$. Player I wins the play $\langle \mathcal{E}_1,k_1,\mathcal{E}_2,k_2,\dots\rangle$ of $G(X)$ iff for each sequence $\langle t_1,t_2,\dots\rangle \in (R^+)^N$ with $\sum t_n<\infty$ and each sequence $\langle x_1,x_2,\dots\rangle \in X^N$ with $x_n\in t_n\cdot E(n,k_n)$ for each $n\in N$, the sum $\sum x_n$ converges.

A locally convex vector space X is said to be a webbed space if there exists an indexed family $\{E(k_1,\dots,k_n)\colon \langle k_1,\dots,k_n\rangle \in N^n,\ n\in N\}$ of absolutely convex subsets of X such that
(a) $\bigcup\{E(k)\colon k\in N\}$ absorbs X,

(b) $\bigcup\{E(k_1,\dots,k_n,k): k\in N\}\subset E(k_1,\dots,k_n)$ and $\bigcup\{E(k_1,\dots,k_n,k): k\in N\}$ absorbs $E(k_1,\dots,k_n)$ for each $\langle k_1,\dots,k_n\rangle\in N^n$ and $n\in N$, and (c) for each $\langle k_1,k_2,\dots\rangle\in N^N$, for each $\langle t_1,t_2,\dots\rangle\in(R^+)^N$ with $\sum t_n<\infty$, and for each $\langle x_1,x_2,\dots\rangle\in X^N$ with $x_n\in t_n\cdot E(k_1,\dots,k_n)$ for each $n\in N$, the sum $\sum x_n$ converges.

Webbed spaces (in French: espaces à réseau) were introduced and studied by M. De Wilde [28] in the connection of the Closed Graph Theorem for linear operators (cf. also [13], p.408, espaces bornant).

Theorem 4.1. Player I has a winning strategy in $G(X)$ iff X is a webbed space.

It is to be observed that Theorem 4.1 is just a game-theoretic interpretation of the definition of webbed spaces. Let us note that a similar characterization can be given for webbed subspaces of a locally convex vector space.

5. Analytic sets

Let X be a T_1 space. We define a game $G(X)$ as follows. Player I chooses a sequence $\mathcal{E}_1=\langle E(1,1),E(1,2),\dots\rangle$ of subsets of X such that $\bigcup\mathcal{E}_1=X$. After that player II chooses a $k_1\in N$. Assume that $\mathcal{E}_1,k_1,\dots,\mathcal{E}_n,k_n$ have been chosen. Then player I chooses a sequence $\mathcal{E}_{n+1}=\langle E(n+1,1),E(n+1,2),\dots\rangle$ of subsets of X such that $\bigcup\mathcal{E}_{n+1}=$ $=E(n,k_n)$. After that player II chooses a $k_{n+1}\in N$. Player I wins the play $\langle\mathcal{E}_1,k_1,\mathcal{E}_2,k_2,\dots\rangle$ of $G(X)$ iff $\bigcap\{E(n,k_n): n\in N\}=\{x\}$ for some $x\in X$ and for each open set G in X with $G\supset\bigcap\{E(n,k_n): n\in N\}$ we have $G\supset E(n,k_n)$ for some $n\in N$.

Theorem 5.1. Player I has a winning strategy in $G(X)$ iff X is the continuous image of N^N (N^N is treated as the Tihonov product of of $\aleph_0$ copies of the discrete space N).

For the proof of Theorem 5.1 we refer to [27]. From Theorem 5.1 immediately follows

Theorem 5.2. Let X be a nonvoid separable metric space. Then player I has a winning strategy in $G(X)$ iff X is analytic.

Let us note that a similar characterization can be obtained for analytic sets being the images of N^N under upper semi-continuous compact closed-graph correspondences (cf. [10], p.416).

6. Measurable functions

Let f be a real-valued function defined on a set X and let $\mathcal{E}$ be a σ-algebra of subsets of X. We define a game $G(f,X,\mathcal{E})$ as follows. Player I chooses a countable partition $\mathcal{E}_1$ of X such that $\mathcal{E}_1 \subset \mathcal{E}$. After that player II chooses a $E_1 \in \mathcal{E}_1$. Assume that $\mathcal{E}_1, E_1, \ldots, \mathcal{E}_n, E_n$ have been chosen. Then player I chooses a countable partition $\mathcal{E}_{n+1}$ of E_n such that $\mathcal{E}_{n+1} \subset \mathcal{E}$. After that player II chooses a $E_{n+1} \in \mathcal{E}_{n+1}$. Player I wins the play $\langle \mathcal{E}_1, E_1, \mathcal{E}_2, E_2, \ldots \rangle$ of $G(f,X,\mathcal{E})$ iff $\lim d_n = 0$, where $d_n = \sup\{|f(x_1)-f(x_2)| : x_1, x_2 \in E_n\}$ for each $n \in N$.

<u>Theorem 6.1.</u> Player I has a winning strategy in $G(f,X,\mathcal{E})$ iff f is $\mathcal{E}$-measurable.

Theorem 6.1 is an easy consequence of Theorem 6.2 below; the winning strategy can be recognized in the condition (d).

<u>Theorem 6.2.</u> Let f be a real-valued function defined on a measurable space $\langle X, \mathcal{E} \rangle$. Then the following conditions are equivalent:
(a) f is $\mathcal{E}$-measurable.
(b) There exists a countable subfamily $\mathcal{A}$ of $\mathcal{E}$ such that $\{f(A) : A \in \mathcal{A}\}$ is a network in $f(X)$ (i.e., for each $y \in f(X)$ and each open interval J containing y there exists a $A \in \mathcal{A}$ such that $y \in f(A) \subset J$.
(c) There exists a countable subfamily $\mathcal{B}$ of $\mathcal{E}$ such that f is continuous with respect to the topology on X generated by $\mathcal{B}$.
(d) There exists a sequence $\langle \mathcal{E}_1, \mathcal{E}_2, \ldots \rangle$ of countable partitions of X such that for each $n \in N$ we have: $\mathcal{E}_n \subset \mathcal{E}$, $\mathcal{E}_{n+1}$ refines $\mathcal{E}_n$, and for each $E \in \mathcal{E}_n$, $\sup\{|f(x_1)-f(x_2)| : x_1, x_2 \in E\} \leq 1/n$.

Let us note that the condition (d) of Theorem 6.2 is nothing else as the regularity of f relative to the P-system $\langle \mathcal{E}_1, \mathcal{E}_2, \ldots \rangle$ considered by J.Dravecký ([7] and [8]).

7. Metrizable spaces

Let X be a regular space. We define a game $G(X)$ as follows.

Player I chooses a locally finite family $\mathcal{E}_1$ of open sets in X such that $\bigcup\mathcal{E}_1 = X$. After that player II chooses a $E_1 \in \mathcal{E}_1$. Assume that $\mathcal{E}_1, E_1, \ldots, \mathcal{E}_n, E_n$ have been chosen. Then player I chooses a locally finite family $\mathcal{E}_{n+1}$ of open sets in X such that $\bigcup\mathcal{E}_{n+1} = E_n$. After that player II chooses a $E_{n+1} \in \mathcal{E}_{n+1}$. Player I wins the play $\langle \mathcal{E}_1, E_1, \mathcal{E}_2, E_2, \ldots \rangle$ of G(X) iff either $\bigcap\{E_n : n \in N\} = 0$ or there exists a point $x \in X$ for which $\langle E_1, E_2, \ldots \rangle$ is a local base.

Theorem 7.1. Player I has a winning strategy in G(X) iff X is metrizable.

Theorem 7.1 is an immediate consequence of the following.

Theorem 7.2. A regular space X is metrizable iff there exists a sequence $\langle \mathcal{B}_1, \mathcal{B}_2, \ldots \rangle$ of locally finite open covers of X such that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ for each $n \in N$, and if $\langle B_1, B_2, \ldots \rangle$ is a sequence of open sets in X such that $B_{n+1} \subset B_n \in \mathcal{B}_n$ for each $n \in N$, then either $\bigcap\{B_n : n \in N\} = 0$ or there exists a point $x \in X$ for which $\langle B_1, B_2, \ldots \rangle$ is a local base.

Theorem 7.2 is a slight modification of the Nagata-Smirnov Metrization Theorem (cf. [9], p.196). Let us note that a similar characterization can be obtained for completely metrizable spaces, M--spaces (cf. [17]), and for some other relative classes of spaces.

8. P-spaces

Let X be a space. We define a game G(X) as follows. Player I chooses an open set G_1 in X. After that player II chooses a closed set F_1 in X such that $F_1 \subset G_1$. Assume that $G_1, F_1, \ldots, G_n, F_n$ have been chosen. Then player I chooses an open set G_{n+1} in X. After that player II chooses a closed set F_{n+1} in X such that $F_{n+1} \subset \bigcup\{G_k : k \leq n+1\}$. Player II wins the play $\langle G_1, F_1, G_2, F_2, \ldots \rangle$ of G(X) iff either $\bigcup\{G_n : n \in N\} \neq X$ or $\bigcup\{G_n : n \in N\} = X = \bigcup\{F_n : n \in N\}$.

P-spaces were introduced by K.Morita ([16] and [17]) for an intrinsic characterization of those normal (resp. paracompact) spaces X whose product space $X \times Y$ is normal (resp. paracompact) for each metric space Y.

Theorem 8.1. Player II has a winning strategy in G(X) iff X is a P-space.

Theorem 8.1 immediately follows from Theorem 8.2 below.

Theorem 8.2. X is a P-space iff there exists a function F defined on the set of all finite sequences $\langle G_1,\dots,G_n\rangle$ of open sets in X such that $F(G_1,\dots,G_n)$ is a closed set in X, $F(G_1,\dots,G_n)\subset\bigcup\{G_k: k\leq n\}$, and $\bigcup\{F(G_1,\dots,G_n): n\in N\} = X$ for each sequence $\langle G_1,G_2,\dots\rangle$ of open sets in X with $\bigcup\{G_n: n\in N\} = X$.

For the proof of Theorem 8.2 we refer to [26], where a similar characterization was obtained for P(m)-spaces as well.

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