REGULARITY: A GENERALIZATION OF EQUICONTINUITY

SAROOP K. KAUL

Regina

1. Single valued functions:

Let (X,Y) denote the set of all functions from a space X to a space Y. We say $F \subseteq (X,Y)$ is regular at $x \in X$ if given an open set U and a subset $H \subset F$ such that $\overline{H(x)} \subset U$, where $H(x) = \{f(x): f \in H\}$, there exists an open set V containing x such that $H(V) \subset U$, where $H(V) = \bigcup\{H(z): z \in V\}$. F is said to be regular if it is regular at each point of X. For the definition of even continuity see [7].

Theorem (1.1): If Y is a regular space, $F \subset (X,Y)$ and $\overline{F(x)}$ is compact for each $x \in X$, then F is regular if and only if it is evenly continuous.

Thus, as a corollary to this, we have that in the Ascoli theorem in [4] one can replace even continuity by regularity.

Theorem (1.2): Let Y be a regular space, $F \subset (X,Y)$ be regular, and $\overline{F}$ be the pointwise closure of F. Then $f \in \overline{F}$ implies f is continuous.

These and other results have been proved in [3].

2. Set valued functions:

Let us consider the set of all set valued functions $\mathfrak{f} = \mathfrak{f}(X,Y)$, from a space X to a space Y, where a set valued function $f: X \to Y$ induces a single valued function $\hat{f}: X \to 2^Y$, 2^Y being the set of all closed non-empty subsets of Y. Given any topology t on 2^Y, we say that f is t-continuous if $\hat{f}$ is t-continuous; and also talk of the pointwise topology p_t and the compact open topology c_t on $\mathfrak{f}$ as those which make $f \leftrightarrow \hat{f}$ a homeomorphism with the corresponding topologies on $\hat{f} = \{\hat{f}: f \in \mathfrak{f}\} = (X,2^Y)$. Let κ, τ, and ν denote the upper semi-finite, lower semi-finite and finite topologies on 2^Y [9] respectively. It is interesting then that natural generalizations of regularity and even continuity give results, similar to those in §1, for set valued functions. So let $F \subset \mathcal{F}$.

Regularity: F is said to be regular at $x \in X$ if given an open set U in Y and a subset H of F such that $\overline{H(x)} \subset U$, where $H(x) = \cup\{f(x): f \in H\}$, there exists an open set V containing x such that $H(V)$ U, where $H(V) = \cup\{H(x): x \in V\}$.

Even Continuity: F is said to be evenly continuous at $x \in X$ if given any $y \in Y$ and a closed neighbourhood V of y there exists an open neighbourhood W of y and an open neighbourhood U of x, such that if $g \in F$ and $g(y) \cap W \neq \phi$ then $U \subset \tilde{g}[V] = \{z \in X: g(z) \cap V \neq \phi\}$; F is said to be evenly continuous if it is evenly continuous at each point of X.

Let $\mathcal{S} = \{f \in \mathcal{F}: f(x)$ is compact for each $x \in X\} = \mathcal{S}(X,Y)$.

Theorem (2.1): Let Y be a normal space. If $F \subset \mathcal{S}$ is regular, $\overline{F(x)}$ is compact for each $x \in X$, and F is a closed subset of $(\mathcal{S}, c_\kappa)$, then (F, c_κ) is compact.

Theorem (2.2): Let Y be a regular space. If $F \subset \mathcal{S}$ is evenly continuous, $\overline{F(x)}$ is compact for each $x \in X$ and F is closed in $(\mathcal{S}, c_\tau)$, then (F, c_τ) is compact.

Theorem (2.3): Let Y be a regular space, X be locally compact. If F is a closed subset of $(\mathcal{S}, c_\nu)$ then (F, c_ν) is compact if and only if (1) F is regular, (2) F is evenly continuous, and (3) $\overline{F(x)}$ is compact for each $x \in X$.

Remark. For single valued F, under the hypothesis of the theorem, even continuity and regularity are the same. Hence this theorem gives a complete generalization of the Ascoli theorem [7, theorem 21, p. 236], for ν-continuous compact valued functions.

Theorem (2.4): Let Y be a regular space, and $F \subset \mathcal{S}$ be a set of t-continuous functions is $\mathcal{S}$ for a topology t on 2^Y. If $f \in cl_{P_t}(F)$ in $\mathcal{S}$, then f is t-continuous for $t = \kappa$ or $t = \tau$ provided F is respectively regular or evenly continuous.

3. Following Fuller's result [2], for local compactness in single valued function spaces, we have the following theorems for set valued functions. Let $S = S(X,Y)$ be as in §2 above.

Theorem (3.1): Let X be a compact T_2-space and Y be a locally compact T_2-space. If $F \subset S$ is regular and evenly continuous then $\overline{F}$ the closure of F with respect to p_ν in S is locally compact.

Theorem (3.2): Let X be a compact T_2-space and Y be a locally compact normal T_2-space. If $F \subset S$ is regular then $\overline{F}$ the closure of F in (S, p_ν) is locally compact.

4. Regularity:

Let $\mathcal{S} = \mathcal{S}(X,Y)$ be the set of all set valued functions from a space X to a space Y as in §2. Given any $f \in \mathcal{S}$, by the graph $G(f)$ of f is meant the subset $\{(x,y): y \in f(x), x \in X\}$ of $X \times Y$. For any $F \subset \mathcal{S}$ we define the set valued function $\pi_F: X \to Y$ given by $\pi_F(x) = \overline{F(x)}$, $x \in X$. From a theorem of Billera [1] we have immediately,

Theorem (4.1): Let Y be a compact T_2-space. Then $F \subset \mathcal{S}(X,Y)$ is regular if and only if for any subset H of F, π_H has a closed graph in $X \times Y$.

Again, given $f \in \mathcal{S}$ one can write formally $f^*(y) = \{x: y \in f(x)\}$. If f^* is a set valued function from Y to X then $(x,y) \to (y,x)$ maps $G(f)$ onto $G(f^*)$ in 1-1 manner. If $F \subset \mathcal{S}$ set $F^* = \{f^*: f \in F\}$.

Theorem (4.2): Let X and Y be compact T_2-spaces. If $F \subset \mathcal{S}(X,Y)$ and $F^* = \mathcal{S}(Y,X)$ mapping $*: (F, p_\nu) \to (F^*, p_\nu)$ given by $*(f) = f^*$ is a homeomorphism.

Remark 1. In the above theorem F and F* regular implies that each is a set of u.s.c., functions but not necessarily l.s.c.

Remark 2. Let X and Y be compact T_2-spaces and SO(X,Y) be the set of all open and continuous functions from X onto Y.

Then the set $D(Y,X) = \{f^* = f^{-1}: \ f \in SO(X,Y)\}$ is the set of all ν-continuous, open "decompositions" from Y onto X, where f* is a decomposition means that for $y_1, y_2 \in Y$ and $y_1 \neq y_2$, $f^*(y_1) \cap f^*(y_2) = \phi$, and onto means that $f^*(Y) = X$. For single valued functions let c and p denote the compact-open and the pointwise topologies respectively on the function space.

Theorem (4.3): Let $F \subset SO(X,Y)$ and (F,c) be compact. If for any net $\{f_\alpha\}$ in F, $\cup\{\limsup f_\alpha(y): \ y \in Y\} = X$, then $*: \ (F,c) \to (F^*, c_\nu)$ is a homeomorphism.

Theorem (4.4): Let $F \subset D(Y,X)$ and (F,c_ν) be compact. Then $*: \ (F,c_\nu) \to (F^*,c)$ is a homeomorphism.

Corollary to theorem (4.3): Suppose X and Y are compact T_2-spaces and $\{f_n\}$ is a sequence of monotone open mapping from X onto Y converging in c to an open mapping f from X onto Y. If $\cup\{\limsup_{i\to\infty} f^{-1}_{n_i}(y): \ y \in Y\} = X$ for each subsequence $\{f_{n_i}\}$ of $\{f_n\}$, then f is monotone.

Corollary to theorem (4.4): Suppose F is a set of monotone open mappings in SO(X,Y) where X and Y are compact T_2-spaces. If F* has a compact closure in $(D(Y,X),c_\nu)$ then $\overline{F}$ the closure of F in (SO(X,Y),c) is compact and $f \in \overline{F}$ implies that f is monotone.

5. Regularity and Even Continuity:

Theorem (5.1): Let $F \subset S(X,Y)$ be regular and evenly continuous, let X be a compact T_2-space and Y be a regular T_2-space. If $\{f_\alpha\}$ is a net in F converging to $f \in F$ with respect to p_ν, then $\{G(f_\alpha)\}$ converges to $G(f)$ in $(2^{X\times Y},\nu)$.

Theorem (5.2): Let Y be a regular T_2-space and $F \subset S(X,Y)$ be regular and evenly continuous. Let $\{f_\alpha\}$ be a net in F and $\{G(f_\alpha)\}$ converge to a compact subset $A \in 2^{X\times Y}$ with respect to ν on $2^{X\times Y}$. Then $A = G(f)$ for some $f \in S(X,Y)$ and $\{f_\alpha\}$ converges to f with respect to p_ν.

References

[1] Billera, L.J.: "Topologies for 2^X: set valued function spaces and their graphs". Trans. Amer. Math. Soc. 155(1971).

[2] Fuller, R.V.: "Conditions for a function space to be locally compact". Proc. Amer. Math. Soc. 36(1972) 615-617.

[3] Kaul, S.K.: "Compact subsets in function spaces". Can. Math. Bull. 12(1969), 461-466.

[4] __________: "Ascoli theorem and the exponential map in set valued function spaces". Rend. Math. Roma. Vol. 9, See Vl(1976) 165-177.

[5] __________: "Local Compactness in set valued function spaces". Can. Math. Bull. (1976) next issue.

[6] __________: "On Monotone open Mappings". submitted for publication.

[7] Kelley, J.L.: "General Topology". Van Nostrand 1955.

[8] Kuratowski, K.: "On the completeness of the space of monotone mappings and some related problems". Bull. Acad. Polon. Sci. 16(1968) 283-285.

[9] Michael, E.: "Topologies on Spaces of Subsets". Trans. Amer. Math. Soc. 71(1951) 152-182.