Toposym 4-B

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SEQUENTIALLY COMPLETE SPACES

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Sequentially regular convergence spaces, i.e. spaces in which convergence of sequences is projectively generated by classes of functions, and their sequential envelopes have been introduced by J. Novák at the First Prague Symposium [7]. This paper is devoted to the sequential completeness of sequentially regular convergence spaces and extensions of sequentially continuous mappings.

In notation and terminology we generally follow J. Novák [8]. Recall that a convergence space is a closure space (X, λ) (cf. [1]) where the closure operator is induced by a sequential convergence on X, i.e. $\lambda A = \{x \mid x = \lim x_n, \bigcup(x_n) \subset A\}$. Let (X,u) be a closure space. The convergence of sequences in (X,u) is defined in the usual way, i.e. $\langle x_n \rangle$ converges to x iff each neighborhood of x contains x_n for all but finitely many n. The corresponding convergence closure for X will be denoted by λ_u. The space (X,u) is said to be F-sequentially regular if the convergence of sequences in X is projectively generated by $F \subset R^X$, i.e. $\lim x_n = x$ iff $\lim f(x_n) = f(x)$ for each $f \in F$. If $F = C(X)$, then we simply say that X is sequentially regular. A set A is sequentially closed in (X,u) iff $\lambda_u A = A$; it is sequentially dense in X iff $\lambda_u^{\omega_1} A = X$, where $\lambda_u^{\omega_1}$ is the topological modification of λ_u. A subspace (Y,v) of (X,u) is said to be sequentially F-embedded in X if each $f \in F \subset C(Y, \lambda_v)$ has a continuous extension $\bar{f} \in C(X, \lambda_u)$.

<u>Definiton 1.</u> Let X be a closure space and $F \subset R^X$. A sequence $\langle x_n \rangle$ of points of X is said to be <u>F-fundamental</u> if for each $f \in F$ the sequence $\langle f(x_n) \rangle$ converges in R.

<u>Definition 2.</u> Let X be a closure space and $F \subset R^X$. The space X is said to be <u>F-sequentially complete</u> if every F-fundamental sequence converges in X. If $F = C(X)$, then we simply say that X is <u>sequentially complete</u>.

Note that an F-sequentially complete space need not be F-sequentially regular. E.g. if $F = R^X$, then X is always F-sequentially complete but X is F-sequentially regular iff it is discrete.

Definition 3. Let X be a closure space and $F \subset R^X$. The space X is said to have the property p with respect to F if

(p) for every two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of points of X such that $(\lambda_u \cup (x_n)) \cap (\lambda_u \cup (y_n)) = \emptyset$ there is a function $f \in F$ such that $\lim f(x_n) = \lim f(y_n)$ does not hold.

If $F = C(X)$, then we simply say that X has the property p.

This property has been introduced in [3] for convergence spaces and $F \subset C(X)$. It has been studied in [2] in the special case when $F = C(X) \cap E^X$, where $E \subset R$.

Let $C_o \subset C(X)$. The C_o-sequentially complete spaces are characterized by the following theorem, where P-space stands for one of the following: closure space, convergence space, topological space, sequential space.

Theorem 4. Let X be a C_o-sequentially regular P-space. Then the following statements are equivalent.

(i) X is C_o-sequentially complete.

(ii) X has the property p with respect to C_o.

(iii) X is sequentially closed in every P-space in which it is sequentially C_o-embedded.

(iv) X is sequentially closed in every P-space in which it is C_o-embedded.

Corollary 5. Let X be a completely regular space. Then the following statements are equivalent.

(i) X is sequentially complete.

(ii) X has the property p.

(iii) X is sequentially closed in the Hewitt realcompactification νX of X.

(iv) X is sequentially closed in the Čech-Stone compactification βX of X.

The notion of F-sequential completeness generalizes several previous definitions of sequential completeness. For $F = C(X)$ and X sequentially regular convergence space we obtain $\mathcal{L}$-completeness defined in [6]. By Theorem 4 for $F \subset C(X)$ and X F-sequentially regular convergence (sequential) space we obtain C_0-sequential completeness defined in [2]. By Corollary 5 for $F = C^*(X)$ and X completely regular we obtain sequential completeness defined in [5].

Now we shall consider C_0-sequentially regular convergence spaces. Since not all sequentially regular spaces are sequentially complete (see e.g. [8]) it is natural to consider a suitable sequentially complete convergence space into which a given space can be embedded as a sequentially dense subspace. The following definition has been given by J. Novák in [9].

Definition 6. Let (L,λ) be a C_0-sequentially regular convergence space. A convergence space (S,σ) is said to be a C_0-sequential envelope $\sigma_0(L)$ of (L,λ) if

(e_1) (L,λ) is a sequentially dense C_0-embedded subspace of (S,σ),

(e_2) (S,σ) is $\bar{C}_0(S)$-sequentially regular, where $\bar{C}_0(S) = \{ f \in C(S) \mid f|L \in C_0 \}$, and

(e_3) there is no convergence space (S',σ') containing (S,σ) as a proper subspace and satisfying (e_1) and (e_2) with respect to (L,λ).

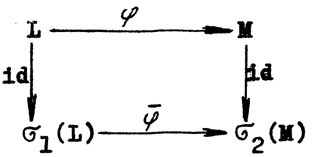
The C_0-sequential envelope is unique in the sense that if S_1 and S_2 are C_0-sequential envelopes of L, the there is a homeomorphism of S_1 onto S_2 that leaves L pointwise fixed (Theorem 5 in [9]) and we write $S_1 = S_2$.

<u>Theorem 7</u>. Let (L, λ) be a C_o-sequentially regular convergence space. Then the following statements are equivalent.

(i) $\sigma_o(L) = (L, \lambda)$.

(ii) (L, λ) is C_o-sequentially complete.

<u>Theorem 8</u>. Let φ be a continuous mapping of a $C_1(L)$-sequentially regular convergence space (L, λ) into a $C_2(M)$-sequentially regular convergence space (M, μ). If $\varphi \circ C_2(M) \subset C_1(L)$, then there is a continuous mapping $\bar{\varphi}$ of $\sigma_1(L)$ into $\sigma_2(M)$ such that $\bar{\varphi}|L = \varphi$ and the diagram



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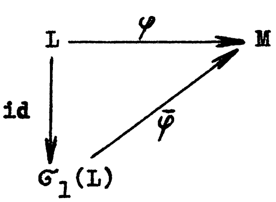
For $M = L$, $\varphi = id$ we obtain Theorem 6 in [9] as a special case.

From Theorem 7 and Theorem 8 we obtain the following

<u>Corollary 9</u>. Let φ be a continuous mapping of a $C_1(L)$-sequentially regular convergence space (L, λ) into a $C_2(M)$-sequentially regular convergence space (M, μ) and let the space (M, μ) be $C_2(M)$-sequentially complete. If $\varphi \circ C_2(M) \subset C_1(L)$, then there is a continuous mapping $\bar{\varphi}$ of $\sigma_1(L)$ into M such that $\bar{\varphi}|L = \varphi$ and the diagram



commutes.

The proofs and details will appear elsewhere [4].

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