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ON COHESIVE MAPPINGS

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In this paper some kind of mappings, named cohesive ones, is introduced. There are a few propositions concerning properties of these mappings and their relations to confluent and atomic mappings. Some characterizations of confluent mappings are obtained.

In the whole paper f denotes a continuous mapping of a topological space X onto a topological space Y.

1. Definitions and preliminary properties. Let us call f cohesive at the point $y \in Y$ if $f^{-1}(y) \subset \overline{f^{-1}(S)}$ for each connected subset S of Y such that $y \in \overline{S}$. We say that f is cohesive, if it is cohesive at every point $y \in Y$. In other words, f is cohesive if and only if there is $f^{-1}(\overline{S}) \subset \overline{f^{-1}(S)}$ (or, equivalently, $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$) for each connected subset S of Y.

Note that in case when f maps a metric irreducible continuum X onto the unit interval Y in such a way that $f^{-1}(y)$, $y \in Y$, are layers of X (for definition see [6], p. 199), then f is cohesive if and only if each layer of X is a layer of cohesion in the sense of Kuratowski (see [6], p. 201).

It is clear in view of [5], p. 117, Theorem 1, that if f is open at the point y (for definition see [5], p. 116), then f is cohesive at y. So the class of all cohesive mappings contains the class of all open mappings. On the other hand it is easy to see that there are cohesive mappings which are not open, for example the projection of the plane curve $\{(x,y):\ y = \sin\frac{1}{x},\ 0 < x \leqslant 1\} \cup \{(x,y):\ x = 0,\ -1 \leqslant y \leqslant 1\}$ on the x axis.

Note that there are monotone mappings which are not cohesive, for example the projection of the plane curve
$\{(x,y):\ y = \sin\frac{1}{x},\ 0 < x \leqslant 1\} \cup \{(x,y):\ x = 0,\ -1 \leqslant y \leqslant 2\}$ on the x axis.

Lemma. The inclusions

$$B \subset Y \setminus f(X \setminus f^{-1}(B)) \subset \overline{B}$$

hold true for each f and each subset B of Y.

Proof. It follows from $f^{-1}(B) \cap X \setminus f^{-1}(B) = \emptyset$ that $B \cap f(X \setminus f^{-1}(B)) = \emptyset$ and consequently $B \subset Y \setminus f(X \setminus f^{-1}(B))$.

On the other hand, it follows from continuity of f that $\overline{ff^{-1}(B)} \subset f\overline{f^{-1}(B)} = \overline{B}$. Thus $f(X) \setminus \overline{B} \subset f(X) \setminus ff^{-1}(B) \subset f(X \setminus f^{-1}(B))$ and consequently $Y \setminus f(X \setminus f^{-1}(B)) \subset \overline{B}$.

Proposition 1. A mapping f is cohesive if and only if $f(U)$ is open in Y for each open subset U of X such that the set $Y\setminus f(U)$ is connected.

Proof. Assume that f is cohesive and U is an open subset of X such that the set $Y\setminus f(U)$ is connected. Let $S = Y\setminus f(U)$. Therefore $f^{-1}(S) = X\setminus f^{-1}f(U)\subset X\setminus U$ and consequently $\overline{f^{-1}(S)}\subset X\setminus U$. Since $f^{-1}(\overline{S})\subset\overline{f^{-1}(S)}$, then $f^{-1}(\overline{S})\cap U = \emptyset$. So $\overline{S}\cap f(U) = \emptyset$. It means that $f(U)$ is open.

Concerning the inverse implication let S be an arbitrary connected subset of Y and let $U = X\setminus\overline{f^{-1}(S)}$. By the lemma we have $S\subset Y\setminus f(U)\subset\overline{S}$, whence $Y\setminus f(U)$ is connected. It implies that $f(U)$ is open in Y. So $\overline{S} = Y\setminus f(U)$ and $f^{-1}(\overline{S}) = X\setminus f^{-1}f(U)\subset X\setminus U = \overline{f^{-1}(S)}$. Thus f is cohesive.

Proposition 2. A mapping f is cohesive if and only if $f(U)$ is open in Y for each open subset U of X such that the family of all components of $Y\setminus f(U)$ is locally finite.

Proof. In view of Proposition 1 it is sufficient to prove only one implication. Then assume that f is cohesive and let U be an open subset of X. Let $\{C_t\}$ denote the family of all components of the set $Y\setminus f(U)$. It follows from $C_t\subset Y\setminus f(U)$ that $f^{-1}(C_t)\subset X\setminus U$ and consequently $\overline{f^{-1}(C_t)}\subset X\setminus U$. Therefore the inclusion $f^{-1}(\overline{C_t})\subset\overline{f^{-1}(C_t)}$ implies $f^{-1}(\overline{C_t})\cap U = \emptyset$. So $\overline{C_t}\cap f(U) = \emptyset$. It means that $C_t = \overline{C_t}$ for every t. Consequently, if the family $\{C_t\}$ is locally finite, then $\overline{\bigcup C_t} = \bigcup\overline{C_t} = \bigcup C_t$. Thus the set $f(U)$ is open.

2. Relations to confluent mappings. In this part the space X is assumed to be compact Hausdorff.

Proposition 3. The following conditions are equivalent:

(1) for every subcontinuum Q of Y each non-empty open-closed subset of $f^{-1}(Q)$ is mapped by f onto Q,

(2) for every subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped by f onto Q,

(3) for every connected subset S of Y each component of $f^{-1}(\overline{S})$ meets $f^{-1}(S)$,

(4) for every connected subset S of Y each component of $f^{-1}(\overline{S})$ meets $\overline{f^{-1}(S)}$,

(5) for every connected subset S of Y the only open-closed subset of $f^{-1}(\overline{S})$ containing $f^{-1}(S)$ is $f^{-1}(\overline{S})$,

(6) for every connected subset S of Y each non-empty open-closed subset of $f^{-1}(\overline{S})$ meets $f^{-1}(S)$.

Recall that the mappings satisfying condition (2) were introduced by J. J. Charatonik in [2] as confluent mappings.

(1) $\Rightarrow$ (2). Let Q be an arbitrary subcontinuum of Y and let C be a component of $f^{-1}(Q)$. Since for compact spaces the components coincide with the quasi-components (see [6], p. 169), then there exists the sequence $U_1, U_2, \ldots$ of open-closed subsets of $f^{-1}(Q)$ such that $\bigcap U_i = C$. We may assume, of course, that $U_{i+1} \subset U_i$ for each i. It is easy to see that $f(C) = f(\bigcap U_i) = \bigcap f(U_i)$. It follows $f(C) = Q$, because by (1) we have $f(U_i) = Q$ for each i.

(2) $\Rightarrow$ (3). If C is a component of $f^{-1}(\bar{S})$, then $f(C) = \bar{S}$ by (2). So $f(C) \cap S \neq \emptyset$ and consequently $C \cap f^{-1}(S) \neq \emptyset$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (5). If U is an open-closed subset of $f^{-1}(\bar{S})$ containing $f^{-1}(S)$, then $\overline{f^{-1}(S)} \subset U$. It follows by (4) that every component of $f^{-1}(\bar{S})$ lies in U. Therefore $U = f^{-1}(\bar{S})$.

(5) $\Rightarrow$ (6). Let U be an arbitrary open-closed subset of $f^{-1}(\bar{S})$. If $U \cap f^{-1}(S) = \emptyset$, then $f^{-1}(S) \subset f^{-1}(\bar{S}) \setminus U$. It follows by (5) that $f^{-1}(\bar{S}) \setminus U = f^{-1}(\bar{S})$. So the set U must be empty.

(6) $\Rightarrow$ (1). Let us suppose in the contrary that $Q \setminus f(U) \neq \emptyset$ for some subcontinuum Q of Y and some non-empty open-closed subset U of $f^{-1}(Q)$. Let S be a component of $Q \setminus f(U)$. Then $\bar{S} \cap f(U) \neq \emptyset$ (see [6], Theorem 2, p. 172), whence $f^{-1}(\bar{S}) \cap U \neq \emptyset$. Denote $V = f^{-1}(\bar{S}) \cap U$. It follows from (6) that $V \cap f^{-1}(S) \neq \emptyset$. This is impossible in view of $f^{-1}(S) \subset f^{-1}(Q) \setminus f^{-1}f(U) \subset f^{-1}(Q) \setminus U \subset f^{-1}(Q) \setminus V$. So the last implication is also proved.

Note that in the proofs of all these implications, except the first and the last, the assumption of compactness of X was not used and then it may be omitted. On the other hand one can show that in implications (1) $\Rightarrow$ (2) and (6) $\Rightarrow$ (1) it is essential.

It is clear that (4) holds true for every cohesive mapping f. So we have following

Corollary. If f is cohesive, then f is confluent.

3. Relations to atomic mappings. In this part the space X is assumed to be a Hausdorff continuum. Recall that f is atomic, if for each subcontinuum K of X such that $f(K)$ contains more than one point there is $f^{-1}f(K) = K$ (see [1], [3] and [4]). It was proved in [4] (see Theorem 1) that each mapping f satisfying this condition is monotone.

Proposition 4. If a mapping f is atomic, then f is cohesive.

Proof. Assume that f is atomic and let S be a connected subset of Y. Thus $\overline{f^{-1}(S)}$ is a continuum and consequently we have $\overline{f^{-1}(S)} = f^{-1}f\overline{f^{-1}(S)} = f^{-1}\overline{ff^{-1}(S)} = f^{-1}(\bar{S})$. So f is cohesive.

Proposition 5. If X is an irreducible metric continuum, Y is a closed segment of reals and f is monotone and cohesive, then f is atomic.

Proof. Let X be a metric continuum irreducible between the points a and b. Then $p = f(a)$ and $q = f(b)$ are the end points of Y (see [6], p. 192, Theorem 3). Let K be an arbitrary subcontinuum of X such that f(K) contains more than one point, i.e. f(K) is the closed segment [r,s] with the end points $r < s$ lying between p and q. Thus $f^{-1}([p,r]) \cup K \cup f^{-1}([s,q])$ is a continuum joining a and b. Consequently it is equal to the whole X. Therefore $f^{-1}(\mathrm{Int}[r,s]) \subset K$, whence $\overline{f^{-1}(\mathrm{Int}[r,s])} \subset K$. Since f is cohesive, then we have $\overline{f^{-1}(\mathrm{Int}[r,s])} = f^{-1}([r,s])$. So $f^{-1}f(K) = K$.

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