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Some results on distance functions

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [371]--380.

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1. Introduction: For a topological space, metrizability is a highly desirable property, for the existence of a such a distance-function gives one a valuable tool for proving theorems about the space. For similar reasons and motivated by the possibility of applications (e.g. in topological algebra, functional analysis or the theory of statistical metric spaces, to name only a few), the concept of a metric d on a set X has been generalized in several ways:

(i) by weakening or omitting some of the "classical" metric axioms (pseudo-, quasi-, semimetrics, symmetrics); and
(ii) by generalizing the range of d, i.e.: by considering distance-functions d on X taking their values in a partially or totally ordered set S with some additional structure ("non-numerical distance-functions").

\- Both ways, as well as combinations of these possibilities have been studied by many authors (compare e.g. the - highly incomplete-bibliography and the introduction of [14]). In this paper, we shall deal with not necessarily symmetric distance-functions taking their values in ordered semigroups S and satisfying the triangular inequality.

Studying general distance-functions d usually raises two fundamental questions:
(i) what is the "best" topological structure on X associated with d, beeing appropriate for a given concrete problem; and
(ii) under what conditions is a given topological structure on X induced by a distance-function of a give type? (metrization theorems).

This paper is concerned with some questions of the second type (the first question will be studied extensively in a forthcoming paper).

2. S-metrics: a totally ordered abelian semigroup $(S,+)$ with zero-element 0 is an 0^+-semigroup if $0 = \min S$; $a < b \Rightarrow a + c < b + c$ for all $a,b,c \in S$; and for each $s > 0$, there exists $t > 0$ with $0 < t < s$. The cofinality of S (cof S) is the smallest ordinal ω_μ such that there exists a decreasing ω_μ-sequence $\{s_i / i < \omega_\mu\}$ converging to $0 \in S$ with respect to the order topology of S.

Definition: Let (X,τ) be a topological space and S an 0^+-semigroup, then X is quasimetrizable over S iff there is a function $d: X^2 \to S$ satisfying

(i) $d(x,y) = 0$ iff $x = y$;

(ii) $d(x,z) \leq d(x,y) + d(y,z)$, for all $x,y,z \in X$;

and such that, for each $x \in X$, the system of all balls $B(x,s) = \{y \in X \mid d(x,y) < s\}$, $s \in S \setminus \{0\}$, is a local base for x. (X,τ) is metrizable over S if, moreover, d is a symmetric function. If S is the positive cone of a totally ordered abelian group G with cof $G = \omega_\mu$ we speak of "ω_μ-(quasi) metrizability".

A space X is (quasi) metrizable in the usual sense iff it is ω_0-(quasi) metrizable; this follows, for example, from A.H. Frink's metrization theorem and from H. Ribeiro's quasimetrization theorem, respectively ([14]). – Now we can ask which part of the theory of (quasi) metrizable spaces carries over to S-(quasi) metrizable spaces and which metrization theorems have analogues for this more general theory? Working with such S-metrics generally leads to situations completely different to the classical case. Important differences are caused by the fact that, on the one hand side, cof S may be $> \omega_0$, or, on the other side, a totally ordered semigroup S need not be a topological semigroup with respect to the order topology on S. Here, an 0^+-semigroup S will be called "continuous" if, for every net $\{x_i \mid i \in I\}$ converging to 0 w.r.t. the order topology, the net $\{x_i + x_i \mid i \in I\}$ converges to 0, too. It turns out that continuity of S (in this weak sense) is the crucial point in studying S-metrizable spaces. [1)] Note, for example

[1)] If S is the positive cone of a totally ordered abelian group G then S is, even stronger, a topological semigroup. Therefore, in this case, the theory of S-metrizable spaces (i.e. ω_μ-metrizable spaces) shows strong similarities with the "classical" theory of metric spaces. (ω_μ-metrizable spaces have been studied by many authors see e.g. the literature lists and introductions in [14], [16], [7] and the remark after theorem 7). – For examples showing how "far" a space (quasi) metrized over an arbitrary 0^+-semigroups S can be from ω_μ-(quasi) metrizability, see [17], [14].

that for an arbitrary set X and a distance function $d: X^2 \to S$ over a non-continuous 0^+-semigroup S, the system of all balls $B(x,s)$ need not be a base for a topology on X. - In the following, we collect some S-(quasi) metrizability theorems for general 0^+-semigroups S as well as for continuous 0^+-semigroups S and discuss several applications. - (Compare [14] and [18]).

At first, the following theorems show that for studying S-metrizable spaces, we can restrict ourselves to only three types of 0^+-semigroups.

Example: For an arbitrary initial ordinal α, fix a set $M=\{x_i/i<\alpha\}$ inversely well-ordered by $x_i < x_j \Leftrightarrow i > j$ and add an element $0 < x_i$ $(i \in \alpha)$. Let $S_\alpha(M) =: S_\alpha$ denote the free abelian semigroup over M, i.e. the set of all finite "formal sums" $\Sigma \lambda_i x_i$, $x_i \in M$, λ_i a natural number, with the usual addition. Identify the empty "word" with $0 \in M$. Now, for different elements $s = \Sigma \lambda_i x_i$, $t = \Sigma \mu_i x_i$, where $\Sigma \lambda_i \neq \Sigma \mu_i$, let $s < t$ iff $\Sigma \lambda_i < \Sigma \mu_i$; otherwise, if $\Sigma \lambda_i = \Sigma \mu_i$, take $j = \min\{i | \lambda_i \neq \mu_i\}$ and let $s < t$ iff $\lambda_j < \mu_j$. Then S_α is an 0^+-semigroup with $\operatorname{cof} S_\alpha = \operatorname{cof} \alpha$ **which certainly is non-continuous.**

Theorem 1: A topological space is (quasi) metrizable over a non-continuous 0^+-semigroup S with $\operatorname{cof} S = \omega_\mu$ iff it is (quasi) metrizable over a semigroup S_{ω_μ}.

Theorem 2: For a topological space (X,τ) are equivalent:

(i) τ is (quasi) metrizable over a continuous 0^+-semigroup S with $\operatorname{cof} S = \omega_\mu$;

(ii) τ is ω_μ-(quasi) metrizable;

(iii) τ is (quasi) metrizable over $\mathbb{R}$, if $\mu = 0$, or, for $\mu > 0$, τ is (quasi) metrizable over the lexicographically ordered abelian group $G_{\omega_\mu} = \Pi\, \mathbb{Z}_i$ $(i < \omega_\mu)$, i.e.: the direct product of copies $\mathbb{Z}_i$ of the additive group of the integers $\mathbb{Z}$.

Note that the proofs for non-symmetric distance functions generally are completely different from the analogous proofs for the symmetric case. They are essentially based on theorem 5. (For symmetric distances, (ii) $\Leftrightarrow$ (iii) was known before).

3. Metrization theorems: A topology τ on X is ω_μ-additive iff the intersection of fewer than ω_μ many open sets is open again. - A point $p\in X$ is of cofinality ω_μ if p has a totally ordered local base whose cofinality is ω_μ. (X,τ) is of characteristic ω_μ if every non-isolated point p is of cofinality ω_μ. By convention, a discrete space is of characteristic ω_μ for every ω_μ. We note that "x is of cofinality ω_μ" is equivalent to "x has a totally ordered local base, and x can be gotten as an intersection of ω_μ open sets, and no fewer than ω_μ open sets". - Moreover, we let $\mathrm{char}(p)=\min\{\mathrm{card}\,\mathfrak{U}(p)/\mathfrak{U}(p)$ is a local base of $p\}$.

Theorem 3: For a topological space (X,τ) are equivalent

(i) τ is quasimetrizable over an 0^+-semigroup S with cof $S=\omega_\mu$;

(ii) (X,τ) is a T_1-space of characteristic ω_μ;

(iii) (X,τ) is a ω_μ-additive T_1-space and for every non-isolated point p, char $(p)=\omega_\mu$.

Corollary: Every first countable space is quasimetrizable over a countable 0^+-semigroup (e.g., over S_{ω_0}).

For symmetric distance-functions, we obtain:

Theorem 4: (X,τ) is metrizable over an 0^+-semigroup S with cof $S=\omega_\mu$ iff for every non-isolated point $p\in X$, $\mathrm{char}(p)=\omega_\mu$ and every point has a local base $\mathfrak{U}(p)=\{U_i(p)/i<\omega_\mu\}$ $i>j\Rightarrow U_i(p)\subset U_j(p)$, and such that $q\in U_i(p)\Rightarrow p\in U_i(q)$ for all $p,q\in X$ and $i<\omega_\mu$.

Now we state some results concerning continuous 0^+-semigroups S:

Definition: A system $\mathfrak{C}$ of open sets in (X,τ) is a Q-collection if $\bigcap\{C\in\mathfrak{C}\,|\,x\in C\}$ is open for each $x\in X$. An open base $\mathfrak{B}$ for τ is an ω_μ-Q-base if it is the union of ω_μ many Q-collections $\mathfrak{C}_i$: $\mathfrak{B}=\bigcup\mathfrak{C}_i\,(i<\omega_\mu)$ and, moreover, the intersection of fewer than ω_μ many basis sets in $\mathfrak{B}$ is open. For $\mu=0$, this concept coincides with P. Fletcher's and W.F. Lindgren's σ-Q-bases ([4], compare also S. Nedev [10]). - Here it is worthwhile to mention the related concept of orthobases defined by P. Nyikos and studied e.g. in [11] and [9].

<u>Theorem 5</u>: For a T_1-space (X,τ) are equivalent:

(i) τ is quasimetrizable over a continuous 0^+-semigroup S with cof $S = \omega_\mu > \omega_0$ (i.e.: τ is ω_μ-quasimetrizable by theorem 2);

(ii) τ has a ω_μ-Q-base $(\mu > 0)$;

(iii) for every non-isolated point $p \in X$, char $(p) = \omega_\mu$ $(\mu > 0)$ and every p has a local base $\mathfrak{U}(p) = \{U_i(p) \mid i < \omega_\mu\}$, $U_i(p)$ open, $i > j \Rightarrow U_i(p) \subset U_j(p)$, such that $q \in U_i(p) \Rightarrow U_i(q) \subset U_i(p)$, for all $i < \omega_\mu$;

(iv) τ is non-archimedeanly ω_μ-quasimetrizable, i.e.: there is an ω_μ-quasimetric for τ such that $d(x,y) \leq \max\{d(x,z), d(z,y)\}$, for all $x,y,z \in X$, $(\mu > 0)$;

(v) τ is non-archimedeanly quasimetrizable over an (arbitrary) 0^+-semigroup S with cof $S = \omega_\mu > \omega_0$.

<u>Remark</u>: For $\mu = 0$, H. Ribeiro proved that τ is $(\mathbb{R}-)$quasimetrizable iff every non-isolated point p has a totally ordered countable local base $U_i(p)$, $i = 1,2,3,\ldots$, such that $q \in U_{i+1}(p) \Rightarrow U_{i+1}(q) \subset U_i(p)$.
P. Fletcher and W.F. Lindgren [4], as well as S. Nedev [10], showed that for every T_1-space, the existence of an ω_0-Q-base is sufficient for quasimetrizability (over $\mathbb{R}$) of τ. However, J. Kofner [8] showed that this condition is not necessary. By theorem 5, the situation is different for $\mu > 0$. - Moreover, for $\mu = 0$, P. Fletcher and W.F. Lindgren showed that τ is non-archimedeanly quasimetrizable iff there is a σ-Q-base for τ. Thus a space has a ω_μ-Q-base $(\omega_\mu \geq \omega_0)$ for its topology iff it is non-archimedeanly quasimetrizable over a continuous 0^+-semigroup.

There is another characterization of spaces X having an ω_μ-Q-base for their topology, including a characterization of spaces admitting a σ-Q-base:

Let $S_\mu = \{x_i / i < \omega_\mu\}$, $x_i > x_j \Leftrightarrow i < j$, be an inversely wellordered set and add an element $0 < x_i$, for all $i < \omega_\mu$. Define $x_i + x_j = \max\{x_i, x_j\}$ and $0 + x_i = x_i$, for all $i < \omega_\mu$; then S_μ is a totally ordered abelian semigroup satisfying $a \leq b \Rightarrow a + c \leq b + c$, for all $a,b,c \in S_\mu$.

<u>Theorem 6</u>: A topological space X has an ω_μ-Q-base $\mathfrak{B}$ $(\mu \geq 0)$ for its topology τ if and only if τ can be generated by a quasimetric $d: X \times X \to S_\mu$ over a semigroup S_μ.

Corollary: X is non-archimedeanly quasimetrizable over an 0^+-semigroup S iff it is quasimetrizable over a semigroup S_μ, $\mu \geq 0$.

Theorem 5 has another interesting corollary, too:

Corollary: For a topological space X, let q(X) be the smallest cardinal $\mathfrak{m}$ such that X has a base consisting of a union of $\mathfrak{m}$ Q-collections. Obviously, $q(X) \leq m(X) \leq w(X)$ where m(X) denotes the metrizability degree of X ([7]) and w(X) its weight. Then:

(1) For $\omega_\mu > \omega_o$, X is ω_μ-quasimetrizable iff X is an ω_μ-additive T_1-space with $q(X) \leq \omega_\mu$. ([18]).

(2) Every ω_μ-additive space X with $w(X) \leq \omega_\mu$ is ω_μ-quasimetrizable (Here, $\omega_\mu \geq \omega_o$). ([18]).

(2) generalizes a well known theorem which states that every second countable space is quasimetrizable. (For more details, see [18]). - A detailed study of ω_μ-Q-bases and the cardinality-function q(X) will appear elsewhere.

The following theorem concerns symmetric distance-functions again:

Theorem 7: For every T_1-space (X, τ) are equivalent:

(i) τ is metrizable over a continuous 0^+-semigroup S with cof $S = \omega_\mu > \omega_o$;

(ii) for every non-isolated point $p \in X$, char $(p) = \omega_\mu$ $(\mu > 0)$ and every p has a local base $\mathfrak{U}(p) = \{U_i(p) / i < \omega_\mu\}$, $U_i(p)$ open, $i > j \Rightarrow U_i(p) \subset U_j(p)$, such that $U_i(p) \cap U_i(q) \neq \emptyset \Rightarrow U_i(p) = U_i(q)$;

(iii) X and X^2 are of characteristic $\omega_\mu > \omega_o$ and both spaces have a base $\mathfrak{B}$ of rank 1 (i.e.: $A \cap B \neq \emptyset \Rightarrow A \subset B$ or $A \supset B$ for all $A, B \in \mathfrak{B}$). [11], [2].

Remark: In the theory of ω_μ-metrizable spaces all famous well-known metrization theorems have their analogues. In [12] (and [16]) it is shown that most of them could be derived from an interesting generalization of J. Nagata's general metrization theorem and a universal imbedding theorem for ω_μ-metrizable spaces where $\mu > 0$. Compare also [14], [7] and its bibliographies.

To emphazise the rôle continuity of S (in our weak sense) plays in the theory of S-metrizable spaces we state another metrization theorem: a Čech-uniformity $\mathfrak{U}$ on X (called "semiuniformity" in Čech's book [1]; but note that this expression is used in a different sense, too) is a generalization of A. Weil's diagonal uniformities which comes from dropping the condition that for every $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V \circ V \subset U$. - Obviously every S-metric d on X induces a Čech-uniformity $\mathfrak{U}_d$ in the usual sense. By combining methods developped for proving the preceeding theorems and using a theorem of Stevenson and Thron, we obtain:

Theorem 8: (1) a separated Čech-uniformity $\mathfrak{U}$ on X has a totally ordered base of least cardinality ω_μ iff $\mathfrak{U}$ is induced by an S-metric d on X over an 0^+-semigroup S with cof $S = \omega_\mu$, [16];
(2) Such a Čech-uniformity $\mathfrak{U}$ is a diagonal-uniformity in the sense of Weil iff $\mathfrak{U}$ admitts an S-metric d on X over an continuous 0^+-semigroup S. [16].

4. Some applications: Methods developped in the realm of S-metrizable spaces have many applications. Here, we collect only examples of two types: characterizations of ($\mathbb{R}$-)metrizable spaces and one example showing how certain aspects of S-metrizability could crop up unexpectedly in theories which, for the first glance, seem to belong to completely different realms.

Proposition 9: A topological space X is metrizable iff it is S-metrizable over a continuous 0^+-semigroup S such that there is an element $s \in S$ and every subset $T \subset S$ containing an element $t \leq s$ has a greatest lower bound.

In [17], three examples show that these - in fact independant - properties: the "topological" one of continuity of S (in our weak sense), the "algebraic" one that $a < b \Rightarrow a + c < b + c$, for all $a, b, c \in S$, and the ordertheoretic one stated above are needed essentially to establish the "classical" theory of metrizable spaces. Both papers, [17] and [18], are generally concerned with an investigation between structural properties of a totally ordered abelian semigroup S and (topological) properties of various topological structures induced by S-metrics d on a set X. - Other results obtained in the realm of S-metrizable spaces are, for example:

<u>Theorem 10</u>: ([16]) (1) A T_1-space X is a compact metric space iff X admitts a unique uniformity $\mathfrak{U}$ and $\mathfrak{U}$ has a linearly ordered base;

(2) X is seperable metric space iff X admitts a totally bounded uniformity with a linearly ordered base.

By using results of [13], and theorems of Venkataraman, Rajagopalan and Soundararajan [Gen.Top. 2 (1972), 1-10] and H. Herrlich [6] we obtain a characterization of topologically orderable groups (G,τ), i.e.: there is a linear order $<$ on G such that τ is homeomorphic with the order topology induced by $<$ on G.

<u>Theorem 11</u>: A topological group (G,τ) is topologically orderable iff either it contains an open normal subgroup homeomorphic with the additive group $\mathbb{R}$ of the reals with the usual topology, or τ is non-archimedeanly metrizable over a continuous 0^+-semigroup S.

5. <u>Metrizability over partially ordered groups</u>: Recall that a metric d on X over a totally ordered abelian group G induces a uniformity $\mathfrak{U}_d$ on X by letting the system of all $U_N = \{(x,y) \mid d(x,y) \in N$, N an open neighbourhood of $0 \in G\}$, be a base for $\mathfrak{U}$. - Now let $(I,\leq)$ be a partially ordered index-set, and $H = \Pi\, \mathbb{R}_i$ $(i \in I)$ the direct product where every $\mathbb{R}_i$ is a copy of the additive group of the reals with the usual topology. Equip H with the product topology. Finally, H can be partially ordered as follows: $g = (g_i) < (h_i) = h$ in case, if $g_j > h_j$ for some $j \in I$, there is some k such that $k < j$ and $g_k < h_k$ and $g_m \leq h_m$ for all $m < k$. Now, let $d: X^2 \to H$ be a quasimetric over H (in the sense of § 1), then d induces a quasiuniformity $\mathfrak{U}$ on X by letting $U_N = \{(x,y) \mid d(x,y) \in N$, N an open neighbourhood of $0 \in H\}$ be a base for $\mathfrak{U}$. Obviously, $\mathfrak{U}$ then induces a T_1-topology on X compatible with d. By using, amongst others, known theorems of A. Császár, I.L. Reilly and G. Kalisch, we can show a converse, too:

<u>Theorem 12</u>: For an arbitrary topological space (X,τ) are equivalent:

(i) (X,τ) is a T_1-space;

(ii) (X,τ) is quasimetrizable over a partially ordered group $H = \Pi\, \mathbb{R}_i$ $(i \in I)$ as described above.

Remark: As a consequence of a theorem of G. Kalisch [Bull. AMS 52 (1946), 936-939] it follows that, in theorem 12, τ is completely regular iff there is a symmetric distance-function d ("metric") on X over H generating τ in the above described sense. (Note that Kalisch's theorem was proved also by M. Antonovskii, V. Boltjanskii, T. Sarymsakov in [Metric spaces over semifields (Russ.); Trudy Tashkent.Gos.Univ. No. 191 (1961), 72pp; MR 28, # 1583]).

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